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WALDHAUSEN'S CLASSIFICATION THEOREM FOR FINITELY UNIFORMIZABLE 3-ORBIFOLDS

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ABSTRACT. We define a map between two orbifolds. With respect to this map, we generalize 3-manifold theory to 3-orbifolds. As the main goal, we generalize the Waldhausen's classification theorem of Haken 3-manifolds to finitely uniformizable 3-orbifolds. For applications of the developed theory, we introduce an invariant for links and tangles by using the orbifold fundamental group. With the invariant, we classify a class of links and show the untangling theorem.

INTRODUCTION

The concept of orbifolds is introduced by I. Satake [S] and renamed by W. P. Thurston (Chapter 13 of [Th]). It is a generalization of the concept of manifolds. An n-orbifold is a topological space locally modelled on (an open set in $\mathbf{R}^n$)/ (a finite group action) and each point of it is provided with an isotropy data. A manifold is regarded as an orbifold whose local group on each point is trivial. $|X|$ means the underlying space of the orbifold X. ΣX means the set of all singular points of local groups of X. In §1, we introduce the basic facts about orbifolds (boundary, orientation, compactness, etc).

A covering orbifold $\widetilde{X}$ of an orbifold X is an orbifold which can be mapped onto X and is locally the quotient by a subgroup of a local group of X. In case that ΣX has codimension greater than 2, orbifold coverings of X are regarded as a branched coverings. There exists a unique universal covering orbifold as a usual covering space [Th]. An orbifold is called good if it has a covering orbifold which is a manifold. In this paper, orbifolds which we deal with are good orbifolds.

For studying orbifolds, we need a map between orbifolds which respects their orbifold structures. We note that the universal cover of a good orbifold inherits the orbifold structure. Let X and Y be orbifolds. Let $p\colon \widetilde{X} \to X$ and $q\colon \widetilde{Y} \to Y$ be the universal coverings. We introduce an orbi-map between X and Y in §2. By an orbi-map $f\colon X \to Y$, we shall mean a continuous map

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$h: |X| \to |Y|$ with a fixed continuous map $\tilde{h}: |\widetilde{X}| \to |\widetilde{Y}|$ which satisfies the following conditions:

(01) $h \circ p = q \circ \tilde{h}$.

(02) For each $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, there exists a $\tau \in \mathrm{Aut}(\widetilde{Y}, q)$ such that $\tilde{h} \circ \sigma = \tau \circ \tilde{h}$.

(03) $h(|X|)$ is not contained in ΣY.

Namely, f is called an orbi-covering (resp. orbi-isomorphism) if h is a covering (resp. isomorphism) as an orbifold.

We rewrite various facts in the theory of 3-manifolds into 3-orbifolds via the functor (manifolds, continuous maps) $\to$ (orbifolds, orbi-maps). Many important concepts in 3-manifold theory are generalized to 3-orbifolds [B-S 1, 2, D, Th]. In [Ta 1], the author studied 3-orbifolds through OR-maps and developed 3-orbifold theory different from one in this paper. He also showed an orbifold version of Waldhausen's classification theorem different from one in this paper.

According to Thurston, the orbifold fundamental group of X, denoted by $\pi_1(X)$, is defined as the deck transformation group of the universal covering $p: \widetilde{X} \to X$. We rewrite it by using paths in $|\widetilde{X}|$ so that an orbi-map $f: X \to Y$ induces a homomorphism $f_*: \pi_1(X) \to \pi_1(Y)$. For the orbifold fundamental groups, covering orbifolds and orbi-maps, we can require the same results as in the theory of usual covering spaces. (See 2.4–2.9.)

In §3, we restrict our target to finitely uniformizable, compact, orientable 3-orbifolds. An orbifold is called finitely uniformizable, if it has the same orbifold structure as the quotient of a manifold by a finite group action. With this assumption, we apply the equivariant theorems of W. H. Meeks and S. T. Yau [M-Y 1–M-Y 3], so that we get orbifold versions of Dehn's Lemma, Loop Theorem, and Sphere Theorem. Thus, we use "cut and paste methods" in studying 3-orbifolds.

In §§4 and 5, we will do the construction and modifications of orbi-maps. For a given homomorphism $\Psi : \pi_1(X) \to \pi_1(Y)$, we can construct an orbi-map $f : X \to Y$ satisfying $f_* = \Psi$ under some conditions. (See 4.2, 4.4.) Frequently we need to modify an orbi-map. A triangulation of an orbifold X is a triangulation of $|X|$ such that ΣX is suitably included in a subcomplex of it. A simplicial orbi-map $f: X \to Y$ is an orbi-map which maps each simplex of some triangulation of X onto a simplex of some triangulation of Y linearly. For a given orbi-map $f : X \to Y$ and a triangulation K_Y of Y, we can modify f preserving some properties into an orbi-map which is simplicial with respect to some triangulations of X and K_Y. (See 5.1.) Using such a modification, we can show the "transversality theorem" by almost the same way as in the case of 3-manifolds. (See 5.5.)

In §6, we will show "I-bundle theorems". Applying the results of W. H. Meeks and P. Scott [M-S], we can get almost the same results as in the case of 3-manifolds. We need this as the breaking case of the main theorem.

In §7, we show the main theorem. Let $\mathscr{W}$ be the class of all compact, connected, orientable 3-orbifolds which are

(i) finitely uniformizable,

(ii) irreducible,

(iii) sufficiently large,

(iv) in which every turnover (a sphere with three singular points) with nonpositive Euler number is boundary-parallel.

By the recent result of W. D. Dunbar [D], the orbifold M which belongs to $\mathscr{W}$ has a hierarchy. The main result then follows from an analog of Theorem 13.6 of [H]. The main result of this paper is as follows.

7.4. **Theorem.** *Let $M, N \in \mathscr{W}$ and suppose $f : (M, \partial M) \to (N, \partial N)$ is an orbi-map such that $f_* : \pi_1(M) \to \pi_1(N)$ is monic and such that for each component B of ∂M, $(f|B)_* : \pi_1(B) \to \pi_1(C)$ is monic, where C is the component of ∂N containing $f(B)$. Then there exists an orbi-map $g : (M, \partial M) \to (N, \partial N)$ such that $g_* = f_* : \pi_1(M) \to \pi_1(N)$ and either*

(1) *$g : M \to N$ is an orbi-covering,*

(2) *M is an I-bundle over a closed 2-orbifold, there is an orbi-homotopy $f_t : (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$, $f_1 = g$, and $g(M) \subset \partial N$, or*

(3) *Each of M and N is* (a) *or* (b) *in Figure 7.2, and $g|\partial M$ is an orbi-covering.*

If $(f|B) : B \to C$ is already an orbi-covering, we may assume $(f|B) = (g|B)$, and in case (2), *$f_t|B = g_t|B$ for all t.*

Furthermore, we show

7.6. **Theorem.** *Let $M, N \in \mathscr{W}$. Suppose all the components of ∂M and ∂N are incompressible in M. Let $\Psi : \pi_1(M) \to \pi_1(N)$ be an isomorphism which respects the peripheral structure. Then either*

(1) *there exists an orbi-isomorphism $f : M \to N$ which induces Ψ or*

(2) *M is a twisted I-bundle over a closed nonorientable* 2*-orbifold F and N is an I-bundle over a* 2*-orbifold G such that $\pi_1(F) \cong \pi_1(G)$.*

As the corollary of it, we derive

7.7. **Corollary.** *Let $M, N \in \mathscr{W}$. Suppose M and N are closed and $\pi_1(M) \cong \pi_1(N)$. Then M and N are orbi-isomorphic.*

Zimmermann, McCullough and Miller showed similar results under a geometric hypothesis. [Z, M-M].

In §8, we apply our result so that we classify a class of links. Recall that a link (S^3, L) is prime if there is no S^2 in S^3 that separates the component of L, and any S^2 that meets L in two points, transversely, bounds in S^3 one and only one ball intersecting L in a single unknotted spanning arc. Let (S^3, L) be a link and X be the orbifold which satisfies that $\Sigma X = L$ and the orders of every local groups of the components of ΣX are $n \in \mathbf{Z}$, $n \geq 2$. We call such an orbifold X the associated orbifold with weight n of (S^3, L), denoted by $O_{(L,n)}$. We define the n-weighted orbi-invariant of (S^3, L), denoted by $\mathrm{Orb}_n(L)$, by the fundamental group of the associated orbifold with weight n of (S^3, L). We call the link (S^3, L) sufficiently large if the orbifold $O_{(L,n)}$ is sufficiently large for some $n \in \mathbf{Z}$, $n \geq 2$. (This definition does not depend on n.) By applying 7.7, we have the following result.

8.1. **Theorem.** *Suppose (S^3, L) and (S^3, L') are prime and sufficiently large links. (S^3, L) and (S^3, L') are the same link type, if and only if* $\mathrm{Orb}_n(L) \cong \mathrm{Orb}_n(L')$ *for some $n \in \mathbf{Z}$, $n \geq 2$.*

Boileau and Zimmermann show a similar result in [B-Z].

For a tangle (B, t), we define $\mathrm{Orb}_n(t)$ in a manner similar to that of the case of links. We get the following result as an application of 4.2 and 5.5.

8.3. **Theorem.** *Let (B, t) be a tangle. (B, t) is the k-strings trivial tangle if and only if* $\mathrm{Orb}_2(t) \cong A_1 * \cdots * A_k$, *where $A_i \cong \mathbf{Z}_2$ for each i.*

In [Ta 2], the untangling theorem for 2-strings case is showed in another form.

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1. Preliminaries

By an *n-dimensional orbifold* X, we shall mean a Hausdorff space S together with a system $\mathscr{S} = (\{U_i\}, \{\varphi_i\}, \{\widetilde{U}_i\}, \{G_i\}, \{\tilde{\varphi}_{ij}\})$ which satisfies the following conditions:

(1) $\{U_i\}$ is locally finite.

(2) $\{U_i\}$ is closed under finite intersections.

(3) For each U_i, there exist a finite group G_i acting smoothly and effectively on a connected open subset $\widetilde{U}_i$ of $\mathbf{R}^n_+$ and a homeomorphism $\varphi_i : \widetilde{U}_i/G_i \cong U_i$.

(4) If $U_i \subset U_j$, there exist a monomorphism $f_{ij} : G_i \to G_j$ and a smooth embedding $\tilde{\varphi}_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ such that for $g \in G_i$, $x \in \widetilde{U}_i$, $\tilde{\varphi}_{ij}(gx) = f_{ij}(g)\tilde{\varphi}_{ij}(x)$ and the following diagram commutes,

$$\begin{array}{ccc} \widetilde{U}_i & \xrightarrow{\tilde{\varphi}_{ij}} & \widetilde{U}_j \\ {\scriptstyle r_i}\downarrow & & {\scriptstyle r_j}\downarrow \\ \widetilde{U}_i/G_i & \xrightarrow{\varphi_{ij}} & \widetilde{U}_j/G_j \\ {\scriptstyle \varphi_i}\downarrow & & {\scriptstyle \varphi_j}\downarrow \\ U_i & \longrightarrow & U_j \end{array}$$

where φ_{ij} is induced by the monomorphism and the embedding, and r_i's are the natural projections.

Each $\varphi_i \circ r_i : \widetilde{U}_i \to U_i$ is called a *local chart* of the orbifold. $\mathscr{S}$ is called an *atlas* of the orbifold. We call S the *underlying space* of the orbifold X, and denote it also by the symbol $|X|$, we mean the underlying space of the orbifold X. X is said to be *connected* if $|X|$ is connected. X is said to be *compact* if $|X|$ is compact. Let $\mathscr{S}'$ be another atlas as above. $\mathscr{S}$ and $\mathscr{S}'$ are said to be *giving the same orbifold structure to* S, if $\mathscr{S}$ and $\mathscr{S}'$ can be combined consistently to give a larger atlas still satisfying the above conditions. When we deal with an orbifold, we will take a convenient atlas in all the atlases which give the same orbifold structures.

Let $\mathscr{S} = (\{U_i\}, \{\varphi_i\}, \{\widetilde{U}_i\}, \{G_i\}, \{\tilde{\varphi}_{ij}\})$ be an atlas of an orbifold X. An *orientation of* U_i is an orientation of $\widetilde{U}_i$ in which G_i acts as a group of orientation preserving maps. We call $\{U_i\}$ an *orientation of* X, if each U_i is oriented and each embedding $\tilde{\varphi}_{ij} : \widetilde{U}_i \to \widetilde{U}_j$ preserves the orientation.

Let $\mathscr{S} = (\{U_i\}, \{\varphi_i\}, \{\widetilde{U}_i\}, \{G_i\}, \{\tilde{\varphi}_{ij}\})$ be an atlas of an orbifold X and H a subspace of $|X|$. We define the *restriction of* $\mathscr{S}$ *to* H, denoted by $\mathscr{S} | H$, by $(\{H_i\}, \{\varphi_i | H_i\}, \{\widetilde{H}_i\}, \{G_i | \widetilde{H}_i\}, \{\tilde{\varphi}_{ij} | \widetilde{H}_i\})$, where $H_i = H \cap U_i$, $\widetilde{H}_i =$ a component of the inverse image of H_i under the quotient map $\widetilde{U}_i \to U_i$, and $G_i | \widetilde{H}_i$ is the restriction of the action of G_i to $\widetilde{H}_i$. By a *subspace of* X, we shall mean a subspace H of $|X|$ together with $\mathscr{S} | H$. We also call H the *underlying space of the subspace of* X. Connectedness, compactness, and other set-theoretical concepts of a subspace of an orbifold are defined by those of its underlying spaces. We shall use the terminology $A \subset X$ to mean that A is a subspace of X.

Let A be a subspace of X. By the symbol $X - A$, we shall mean the subspace of X whose underlying space is $|X| - |A|$. By the symbol $\mathrm{cl}(A)$, we shall mean the subspace of X whose underlying space is $\mathrm{cl}(|A|)$. A *neighborhood of* A *in* X is a subspace of X whose underlying space is a neighborhood of $|A|$ in $|X|$. Let x be a point in $|X|$. A *neighborhood of* x *in* X is a subspace of X whose underlying space is a neighborhood of x in $|X|$.

Let A and B be subspaces of X. By the symbol $A \cap B$ (*resp.* $A \cup B$), we shall mean the subspace of X whose underlying space is $|A| \cap |B|$ (resp. $|A| \cup |B|$). By the symbol $A = B$ (*resp.* $A \subset B$), we shall mean that $|A| = |B|$ (resp. $|A| \subset |B|$).

A subspace Q of an orbifold X is called an *m-dimensional suborbifold* of X, if $\mathscr{S} | H$ gives an m-dimensional orbifold structure to H, where H is the underlying space of Q and $\mathscr{S}$ is an atlas of X.

Let X be an orbifold. Let x be any point in $|X|$ and $\varphi \circ r : \widetilde{U} \to U$ a local chart of X containing x. The *local group at* x, denoted by G_x, is the isotropy group of any point in U corresponding to x. This is well defined up to isomorphisms. A point $x \in |X|$ is called *generic* if $G_x = \{\mathrm{id}\}$ and *singular* otherwise. The set $\Sigma X = \{x \in |X| \,|\, G_x \neq \{\mathrm{id}\}\}$ is called the *singular set* of X.

Note that ΣX is closed and nowhere dense. An orbifold X is called a *manifold* if $\Sigma X = \emptyset$. (This definition is compatible with the ordinary one.) Throughout this paper we will suppose that $\dim \Sigma X \leq n-2$. A *stratum* of X is a maximal connected component on which the orders of the local groups are constant. By the symbol $\Sigma^{(k)}X$, we shall mean all the collection of k-dimensional strata.

Let X be an n-dimensional orbifold. We call a point $x \in |X|$ a *boundary point* of X if for some local chart of X $\varphi_i \circ r_i : \widetilde{U}_i \to U_i$ containing x, x is corresponding to a point in $\widetilde{U}_i \cap \mathbf{R}^{n-1}$. The set of all the boundary points is called the *boundary* of X. By the symbol ∂X, we shall mean the subspace of X whose underlying space is the boundary of X and structure is given by the restriction of $\mathscr{S}$ to it, where $\mathscr{S}$ is an atlas of X. We call $|X| - |\partial X|$ a *interior* of X and a point $x \in |X| - |\partial X|$ a *interior point* of X. By the symbol $\mathrm{Int}(X)$ (*or* $\overset{\circ}{X}$), we shall mean the subspace of X whose underlying space is $|X| - |\partial X|$ and structure is given by $\mathscr{S}|(|X| - |\partial X|)$. ∂X and $\mathrm{Int}(X)$ are clearly (not necessarily connected) suborbifolds of X. ∂X and $\mathrm{Int}(X)$ are called the *boundary orbifold* of X and the *interior orbifold* of X, respectively. Frequently, we also call them simply *boundary* of X and *interior* of X. X is said to be *closed* if X is compact and $|\partial X| = \emptyset$.

Two orbifolds X and X' are called *isomorphic*, if there exists a homeomorphism $h : |X| \to |X'|$ and for each point $x \in |X|$, there exist an isomorphism f_x from the local group G_x of x to the local group $G'_{h(x)}$ of $h(x)$ and a diffeomorphism $\tilde{h}_x : \widetilde{U}_x \to \widetilde{U}'_{h(x)}$ such that, for any $g \in G_x$, any $z \in \widetilde{U}_x$, $\tilde{h}_x(gz) = f_x(g)\tilde{h}_x(z)$, where $\widetilde{U}_x \to \widetilde{U}_x/G_x \cong U_x$ and $\widetilde{U}_{h(x)} \to \widetilde{U}_{h(x)}/G_{h(x)} \cong U_{h(x)}$ are the local charts. h is called an *isomorphism* from X to X'. By the terminology an *isomorphism* $h : X \to X'$, we shall mean that h is an isomorphism from X to X'. (Automatically, X and X' are isomorphic.) In case both X and X' are manifolds, an isomorphism $h : X \to X'$ means an usual diffeomorphism from X to X'.

An orbifold $\widetilde{X}$ is called a *covering orbifold* of an orbifold X, if there exists a continuous map $p : |\widetilde{X}| \to |X|$ which satisfies the following conditions:

(1) p is onto.

(2) each point $x \in |X|$ has a local chart of X of the form $\widetilde{U}_x \to \widetilde{U}_x/G_x \cong U_x$ such that each point $\tilde{x} \in p^{-1}(x)$ has a local chart of $\widetilde{X}$ of the form $\widetilde{U}_x \to \widetilde{U}_x/G_{x,i} \cong V_{x,i}$ and the following diagram commutes, where $V_{x,i}$ is the component of $p^{-1}(U_x)$ including $\tilde{x}$, $G_{x,i}$ is some subgroup of G_x, and q is the natural projection.

$$\begin{array}{ccccc} & & \widetilde{U}_x/G_{x,i} & \cong & V_{x,i} \\ & \nearrow & \downarrow q & & \downarrow p \\ \widetilde{U}_x & \to & \widetilde{U}_x/G_x & \cong & U_x \end{array}$$

p is called a *covering* from X to $\widetilde{X}$. By the terminology a *covering* $p : \widetilde{X} \to X$, we shall mean that p is a covering from $\widetilde{X}$ to X. (Automatically, $\widetilde{X}$ is a covering orbifold of X.) In case both $\widetilde{X}$ and X are manifolds, a covering

$p : \widetilde{X} \to X$ means a usual covering from $\widetilde{X}$ to X. Note that an isomorphism is a covering.

Let $p : \widetilde{X} \to X$ be a covering. An isomorphism $h : \widetilde{X} \to \widetilde{X}$ for which $p \circ h = h$ is called a *deck transformation* of the covering. by the symbol $\mathrm{Aut}(\widetilde{X}, p)$, we mean the group of all the deck transformations of the covering $p : \widetilde{X} \to X$. (The product of σ, $\tau \in \mathrm{Aut}(\widetilde{X}, p)$ is defined by $(\sigma\tau)(x) = \sigma(\tau(x))$, $x \in |\widetilde{X}|$.)

Let M be a manifold and G a group acting smoothly, effectively, and properly discontinuously on M. By Proposition 13.2.1 of [Th], $|M|/G$ has an orbifold structure. We shall use the terminology M/G to denote this orbifold. (Note that $|M/G| = |M|/G$.) It is clear that M is a covering orbifold of M/G, the natural projection $p : |M| \to |M|/G$ is the covering from M to M/G, and that each element $g \in G$ is a deck transformation of the covering $p : M \to M/G$. Moreover, for any subgroup $G' \subset G$, we may regard the natural projection $|M|/G' \to |M|/G$ as a covering from M/G' to M/G.

A covering orbifold which is a manifold is called a *covering manifold* and the covering is called a *manifold covering*. An orbifold is called *good* if it has some covering manifold and *bad* otherwise. By the terminology *a covering* $p : (\widetilde{X}, \tilde{x}) \to (X, x)$, we shall mean that $p : \widetilde{X} \to X$ is a covering with $x \in |X| - \Sigma X$ and $p(\tilde{x}) = x$.

We call a covering $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ the *universal covering* if for any other covering $p' : (X', x') \to (X, x)$, there is a covering $q : \widetilde{X} \to X'$ such that $p' \circ q = p$. $\widetilde{X}$ is called the *universal covering orbifold* of X. The usual proof of the existence and uniqueness of the universal covering can be adapted to show that any orbifold has a (unique) universal covering.

Let X be a good orbifold and M be the universal cover of a covering manifold of X. Naturally M is a covering orbifold of X. Moreover, M is the universal covering orbifold of X since every covering orbifold is isomorphic to M/G', for some $G' \subset G$, where G is the group of the deck transformations of the covering $p : M \to X$. (Note that in this case the universal covering is unique in the sense that for any other universal covering $q : N \to X$ there is an isomorphism (diffeomorphism) $h : M \to N$ such that $p = q \circ h$.) The universal covering determines the orbifold structure of the good orbifold X.

A covering $p : \widetilde{X} \to X$ is called a *regular covering* if for any two preimage $\widetilde{*}$ and $\widetilde{*}'$ of the base point $* \in |X| - \Sigma X$ under p, there exists a deck transformation of the covering taking $\widetilde{*}$ to $\widetilde{*}'$. $\widetilde{X}$ is called a *regular covering orbifold*.

A regular manifold covering $p : \widetilde{X} \to X$ is called a *uniformization* and $\widetilde{X}$ is called a *uniformization manifold* of X.

Let $(\{U_i\}, \{\varphi_i\}, \{\widetilde{U}_i\}, \{G_i\}, \{\widetilde{\varphi}_{ij}\})$ and $(\{V_i\}, \{\psi_i\}, \{\widetilde{V}_i\}, \{H_i\}, \{\widetilde{\psi}_{ij}\})$ be atlases of orbifolds X and Y, respectively. *A product orbifold of* X *and* Y, denoted by $X \times Y$, is an orbifold whose underlying space is $|X| \times |Y|$ and structure is given by $(\{U_i \times V_k\}, \{\varphi_i \times \psi_k\}, \{\widetilde{U}_i \times \widetilde{V}_k\}, \{G_i \times H_k\}, \{\widetilde{\varphi}_{ij} \times \widetilde{\psi}_{kl}\})$.

Let X be an n-orbifold and Y an $(n-1)$-suborbifold of X. Y is said to be *properly embedded* in X, if $\mathrm{Int}(Y) \subset \mathrm{Int}(X)$ and $\partial Y \subset \partial X$. Y is said to be 2-*sided* in X, if there is a suborbifold N of X such that there is an isomorphism $h : Y \times [0, 1] \to N$ with $h(x, 1/2) = x$ for all $x \in |Y|$ and $h(|Y \times [0, 1]|) \cap |\partial X| = h(|\partial Y \times [0, 1]|)$.

Two suborbifolds Y and Z in X are called *ambient orbi-isotropic* in X, if there exists an isomorphism $F : X \times [0, 1] \to X \times [0, 1]$ such that for any $t \in |[0, 1]|$, $F(|X \times t|) \subset |X \times t|$ and $F(|Y \times 0|) = |Z|$ and for any $x \in |X|$, $F(x, 1) = \mathrm{id}$, where $X \times t$ is the suborbifold of $X \times I$ whose underlying space is $|X| \times t$.

We investigate the local behavior of any point $x \in |X|$. Let $\varphi \circ r : \widetilde{U} \to \widetilde{U}/G \cong U$ be a local chart containing x and $\tilde{x}$ be a point in $\widetilde{U}$ corresponding to x. Let $G_{\tilde{x}} < G$ be the isotropy subgroup of $\tilde{x}$ in $\widetilde{U}$ and $U_{\tilde{x}}$ be a $G_{\tilde{x}}$-invariant open neighborhood of $\tilde{x}$ in $\widetilde{U}$. By averaging any Riemannian metric on $\widetilde{U}$ under $G_{\tilde{x}}$, we get a $G_{\tilde{x}}$-invariant metric on $U_{\tilde{x}}$. So the differential dg of $g \in G_{\tilde{x}}$ preserves the inner product which is the restriction of the above metric. This yields a faithful representation $\mu : G_{\tilde{x}} \to O(n)$, well defined up to conjugation in $GL(n, \mathbf{R})$. Consider the exponential map exp which gives a diffeomorphism from the ε-ball $B(0, \varepsilon)$ in $T_{\tilde{x}}U_{\tilde{x}}$ to a small neighborhood $V_{\tilde{x}}$ of $\tilde{x}$ in $U_{\tilde{x}}$. Note that $V_{\tilde{x}}$ is $G_{\tilde{x}}$-invariant. Let f be the natural isometry defined by $f(\sum_{i=1}^{n} a_i\xi_i) = (a_1, \dots, a_n)$, where ξ_i's are a basis for $T_{\tilde{x}}U_{\tilde{x}}$. Since the following diagram commutes, the orbifold $V_{\tilde{x}}/G_{\tilde{x}}$ is isomorphic to $B(0, \varepsilon)/\mu(G_{\tilde{x}})$.

$$\begin{array}{ccccc}
\mathbf{R}^n & & T_{\tilde{x}}U_{\tilde{x}} & & \mathbf{R}^n \\
\supset & & \cup & & \subset \\
B(0, \varepsilon) & \xleftarrow{\ f\ } & B(0, \varepsilon) & \xrightarrow{\ \exp\ } & V_{\tilde{x}} \\
{\scriptstyle O(n) \ni \mu(g)} \downarrow & & \downarrow {\scriptstyle dg} & & \downarrow {\scriptstyle g \in G_{\tilde{x}}} \\
B(0, \varepsilon) & \xleftarrow[\ f\ ]{} & B(0, \varepsilon) & \xrightarrow[\ \exp\ ]{} & V_{\tilde{x}} \\
\supset & & \cap & & \subset \\
\mathbf{R}^n & & T_{\tilde{x}}U_{\tilde{x}} & & \mathbf{R}^n
\end{array}$$

So for any point $x \in |X|$, a neighborhood of x in X is isomorphic to a neighborhood of the origin in the orbifold $\mathbf{R}^n/\Gamma$, where Γ is a finite subgroup of the orthogonal group $O(n)$ which is isomorphic to the local group of x.

We shall consider orientable 2-orbifolds, so we can assume that the local groups are contained in SO(2). The finite subgroups of SO(2) are cyclic groups. The quotient of D^2 by subgroups of SO(2) are cyclic groups. The quotient of D^2 by each of these groups is the topological 2-disc. From this it follows that any orientable 2-orbifold has no codimension 1 strata and must have an orientable 2-manifold as its underlying space. To express each orientable 2-orbifold, we can use a picture of a 2-manifold F together with discrete points p_i, $i = 1, 2, \dots, k$, in $\mathrm{int}(F)$ each of which is assigned integers $n_i \geq 2$. The point p with the integer n means that there is a local chart of the orbifold containing p of the form $\varphi \circ r : \widetilde{U} \to \widetilde{U}/\Gamma \cong U$, where $\widetilde{U}$ is a neighborhood of the origin in $\mathbf{R}^2$, Γ is a cyclic subgroup $\mathbf{Z}_n$ of SO(2), and $p =$ (the image of the origin). We often use the terminology $F(n_1, \dots, n_k)$ to mean such a 2-orbifold.

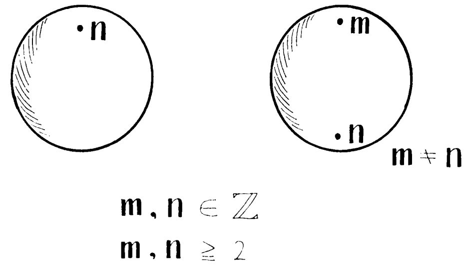


FIGURE 1.1

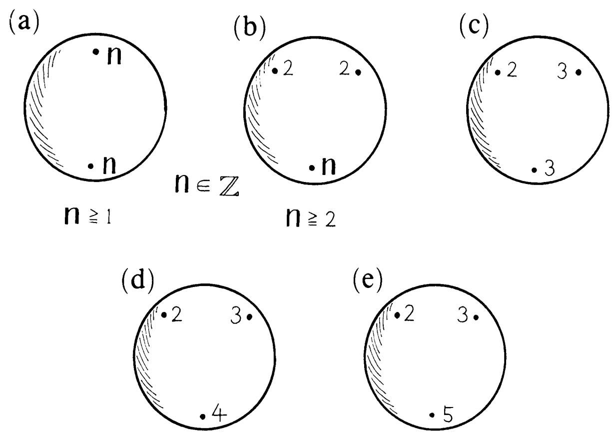


FIGURE 1.2

All the connected, orientable, bad 2-orbifolds are shown in Figure 1.1. None can occur as a suborbifold of any good 3-orbifold, as a consequence of the definitions (of 'good' and 'suborbifold').

All 2-orbifolds with nonempty boundary are good.

All of the orientable good 2-orbifolds whose universal covering manifolds are 2-spheres are shown in Figure 1.2. They are called *spherical orbifolds.* They are isomorphic to S^2/G for some $G \subset \mathrm{SO}(3)$. In each case, G is isomorphic to (a) one of the cyclic groups (b) one of the dihedral groups (c) the tetrahedral group (d) the octahedral group (e) the icosahedral group. These are all the finite subgroups of SO(3).

All of the orientable good 2-orbifolds whose universal covering manifolds are 2-discs are shown in Figure 1.3. They are called *discal orbifolds.* They are isomorphic to $D^2/\mathbf{Z}_n$ for $\mathbf{Z}_n \subset \mathrm{SO}(2)$.

Next, we shall consider orientable 3-orbifolds, so we can assume that the local groups are contained in SO(3). The finite subgroups of SO(3) are listed above. The quotients of D^3 by these groups are shown in Figure 1.4. From this it follows that any orientable 3-orbifold has no codimension 1 strata and must have an orientable 3-manifold as its underlying space. The orbifolds listed in

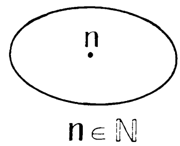


FIGURE 1.3

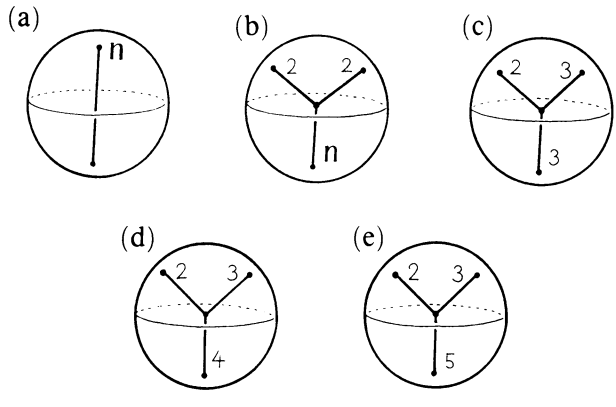


FIGURE 1.4

Figure 1.4 are called *ballic orbifolds*. Note that the boundaries of ballic orbifolds are spherical orbifolds and that the singular set of each ballic orbifold is a cone on its boundary spherical orbifold. In this sense each ballic orbifold is called the cone on the corresponding spherical orbifold. The assigned integer p means that the local group is $\mathbf{Z}_p \subset \mathrm{SO}(3)$. In case (b)–(e), the central vertex is the image of the origin of the local chart, and the only point whose local group is noncyclic. To express each orientable 3-orbifold, we can use a picture of an orientable 3-manifold M together with properly embedded linear graphs $\mathcal{T}$ satisfying the conditions that the valencies of vertices of $\mathcal{T}$ are at most three, each segment l of $\mathcal{T}$ is assigned integers $n \geq 2$, and their local conditions are the same as (a)–(e) in Figure 1.4. The segment l with the integer n means that for each point on l, there is a local chart of the orbifold containing the point of the form $\varphi \circ r : \widetilde{U} \to \widetilde{U}/\Gamma \cong U$, where $\widetilde{U}$ is a neighborhood of the origin in $\mathbf{R}^3$, Γ is a cyclic subgroup $\mathbf{Z}_n$ of SO(2) acting on $\widetilde{U}$ as the rotation with angle $2\pi/n$ around z-axis, and $(l \cap U) =$ (the image of the z-axis). The vertex p to which segments l_1, l_2, l_3 are incident means that there is a local chart of the orbifold containing p of the form $\varphi \circ r : \widetilde{U} \to \widetilde{U}/\Gamma \cong U$, where $\widetilde{U}$ is a neighborhood of the origin in $\mathbf{R}^3$, Γ is either one of the dihedral groups, the tetrahedral group, the octahedral group, or the icosahedral group according to that the triple of the integers n_i assigned to l_i is $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$. Note that the boundaries of orientable 3-orbifolds are orientable closed 2-orbifolds. Throughout this paper, mainly, we deal with orientable 3-orbifolds.

2. Orbi-maps and properties of coverings

From now on, throughout this paper, we assume that orbifolds are good and connected unless otherwise stated. And recall that we supposed all orbifolds have no codimension 1 strata.

Let X be an n-orbifold equipped with a base point $x \in |X| - \Sigma X$, $p : \widetilde{X} \to X$ be the universal covering and $\tilde{x}$ be any lift of x. Put $\Omega(\widetilde{X}, x) = \{\tilde{\alpha} \mid \tilde{\alpha}$ is a continuous map from $[0, 1]$ to $|\widetilde{X}|$ with $p(\tilde{\alpha}(0)) = p(\tilde{\alpha}(1)) = x\}$. Suppose $\tilde{\alpha}, \tilde{\beta} \in \Omega(\widetilde{X}, x)$. Note that there is one and only one element $\iota \in \mathrm{Aut}(\widetilde{X}, p)$ such that $\tilde{\alpha}(0) = \iota(\tilde{\beta}(0))$. $\tilde{\alpha}$ and $\tilde{\beta}$ are called *equivalent*, denoted by $\tilde{\alpha} \sim \tilde{\beta}$, if $\tilde{\alpha}(1) = \iota(\tilde{\beta}(1))$. Clearly this is an equivalence relation in $\Omega(\widetilde{X}, x)$. We use the symbol $[\tilde{\alpha}]$ to denote the equivalence class represented by $\tilde{\alpha}$ in $\Omega(\widetilde{X}, x)/\sim$. Suppose $[\tilde{\alpha}], [\tilde{\beta}] \in \Omega(\widetilde{X}, x)/\sim$. We can define the product of $[\tilde{\alpha}]$ and $[\tilde{\beta}]$, denoted by $[\tilde{\alpha}][\tilde{\beta}]$, by $[\tilde{\alpha} \cdot \rho(\tilde{\beta})]$, where $\cdot$ implies the composition of paths, ρ is the element of $\mathrm{Aut}(\widetilde{X}, p)$ taking $\tilde{\beta}(0)$ to $\tilde{\alpha}(1)$, and $\rho(\tilde{\beta})$ is the path derived from transforming $\tilde{\beta}$ by ρ. It is easy to check that $\Omega(\widetilde{X}, x)/\sim$ becomes a group with this product. So we define $\pi_1(X, x)$, the *fundamental group* of X based on x, by $\Omega(\widetilde{X}, x)/\sim$. Moreover it is easy to show that for any point $\tilde{x} \in p^{-1}(x)$, we can define an isomorphism $\Psi_{\tilde{x}} : \Omega(\widetilde{X}, x)/\sim \to \mathrm{Aut}(\widetilde{X}, p)$ in the following manner. Take a representative path $\tilde{\alpha}$ of $\sigma \in \pi_1(X, x)$ beginning at $\tilde{x}$. Define $\Psi_{\tilde{x}}(\sigma) = \tau$, where τ is the element of $\mathrm{Aut}(\widetilde{X}, p)$ taking $\tilde{\alpha}(0)$ to $\tilde{\alpha}(1)$. So our definition of $\pi_1(X, x)$ respects the ordinary one (Thurston's definition [Th]).

Let $x, x' \in |X| - \Sigma X$ and $\tilde{x} \in p^{-1}(x)$, $\tilde{x}' \in p^{-1}(x')$. Let r be a path from $\tilde{x}$ to $\tilde{x}'$. If $\tilde{\alpha}$ is a path in $|\widetilde{X}|$ such that $\tilde{\alpha}(0) = \tilde{x}$ and $p(\tilde{\alpha}(1)) = x$ then $\tilde{\beta} = r^{-1} \cdot \tilde{\alpha} \cdot \tau(r)$ is a path such that $\tilde{\beta}(0) = \tilde{x}'$ and $p(\tilde{\beta}(1)) = x'$, where τ is the element of $\mathrm{Aut}(\widetilde{X}, p)$ taking $\tilde{\alpha}(0)$ to $\tilde{\alpha}(1)$. We therefore define $u_r : \pi_1(X, x) \to \pi_1(X, x')$ by $u_r[\tilde{\alpha}] = [r^{-1} \cdot \tilde{\alpha} \cdot \tau(r)]$. It is easy to show that u_r is an isomorphism. Note that, if $\tilde{x}''$ is another lift of x' and r' is a path from $\tilde{x}$ to $\tilde{x}''$, then there is an element $\sigma \in \pi_1(X, x')$ such that $u_{r'}[\tilde{\alpha}] = \sigma^{-1} u_r[\tilde{\alpha}]\sigma$, $[\tilde{\alpha}] \in \pi_1(X, x)$. Since $\widetilde{X}$ is connected (equivalently, $|X|$ is pathwise connected), for all $x \in |X| - \Sigma X$, $\pi_1(X, x)$'s are isomorphic. But, note that there are no canonical isomorphisms between them. When we are concerned only with the isomorphism class of the fundamental group of X, we frequently ignore the base point and use the symbol $\pi_1(X)$ to denote $\pi_1(X, x)$ for any chosen $x \in |X| - \Sigma X$.

Let X, Y be orbifolds, and $p : \widetilde{X} \to X$, $q : \widetilde{Y} \to Y$ be the universal coverings. By an *orbi-map* $f : X \to Y$, we shall mean a continuous map $h : |X| \to |Y|$ with a fixed continuous map $\tilde{h} : |\widetilde{X}| \to |\widetilde{Y}|$ which satisfies the following conditions:

(01) $h \circ p = q \circ \tilde{h}$.

(02) For each $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, there exists a $\tau \in \mathrm{Aut}(\widetilde{Y}, q)$ such that $\tilde{h} \circ \sigma = \tau \circ \tilde{h}$.

(03) $h(|X|)$ is not contained in ΣY.

By the continuity of h, (03) is equivalent that there is some point $x \in |X| - \Sigma X$ such that $h(x) \in |Y| - \Sigma Y$.

We call h and $\tilde{h}$, *the underlying map* and *the structure map* of f, respectively. We often use the symbols $\bar{f}$ and $\tilde{f}$ to mean the underlying map and the structure map of the orbi-map $f : X \to Y$, respectively. We often use the terminology $f = (\bar{f}, \tilde{f})$ to mean that the orbi-map f consists of the underlying map $\bar{f}$ and the structure map $\tilde{f}$. Two orbi-maps $f, g : X \to Y$ are equal, denoted by $f = g$, if there exists an element $\tau \in \mathrm{Aut}(\tilde{Y}, q)$ such that $\tilde{f} = \tau \circ \tilde{g}$. (Automatically, $\bar{f} = \bar{g}$.) Note that if we take another universal coverings $p' : \tilde{X}' \to X$ and $q' : \tilde{Y}' \to Y$, then the structure map $\tilde{f}$ changes to $\tilde{h}'^{-1} \circ \tilde{f} \circ \tilde{h}$, where $\tilde{h} : |\tilde{X}'| \to |\tilde{X}|$ and $\tilde{h}' : |\tilde{Y}'| \to |\tilde{Y}|$ are homeomorphisms such that $p' = p \circ \tilde{h}$ and $q' = q \circ \tilde{h}'$. So we identify $\tilde{f}$ and $\tilde{h}'^{-1} \circ \tilde{f} \circ \tilde{h}$ as the structure map of the orbi-map f. It is easy to see that if $\tilde{f}$ is a diffeomorphism from X to Y and that $\bar{f}$ is a homeomorphism from $|X|$ to $|Y|$, then $\bar{f}$ must be an isomorphism from X to Y.

Actually, as maps between orbifolds, several different versions are given [Y, Ta 1]. Here, we give the above definition for the arguments developed in the later parts of this paper.

Let $f : X \to Y$ be an orbi-map. Let A be a subspace of X. By the symbol $f(A)$, we shall mean the subspace of Y whose underlying space is $\bar{f}(|A|)$. Let B be a subspace of Y. By the symbol $f^{-1}(B)$, we shall mean the subspace of X whose underlying space is $\bar{f}^{-1}(|B|)$. We shall use the terminology $f : (X, A) \to (Y, B)$, if $f(A) \subset B$. Let $x \in |X| - \Sigma X$ and $y \in |Y| - \Sigma Y$ be base points. We shall use the terminology $f : (X, x) \to (Y, y)$, if $\bar{f}(x) = y$.

Let $f : X \to Y$ be an orbi-map. Let $p : \tilde{X} \to X$ and $q : \tilde{Y} \to Y$ be the universal coverings. Let P be a suborbifold of X. Note that the inverse image of any suborbifold under a covering has a (not necessarily connected) suborbifold structure of the covering orbifold. Let $\tilde{P}$ be a suborbifold of $\tilde{X}$ whose underlying space is a connected component of $p^{-1}(|P|)$. Let $p' : \tilde{P}' \to P$ be the universal covering. Suppose there is a point $x \in |P| - \Sigma P$ such that $\bar{f}(x) \in |Y| - \Sigma Y$. The map $(\bar{f}|\,|P|) : |P| \to |Y|$ together with the map $(\tilde{f} \circ p') : |\tilde{P}'| \to |\tilde{Y}|$ is said to be the *restriction* of the orbi-map $f : X \to Y$ to P, denoted by $(f\,|\,P) : P \to Y$. Throughout this paper, we assume that, when the symbol $f\,|\,P$ is written for an orbi-map $f : X \to Y$, $P \subset X$, the above conditions are satisfied so that $f\,|\,P$ is definable unless otherwise stated.

We can define a homomorphism $f_* : \pi_1(X, x) \to \pi_1(Y, \bar{f}(x))$ by $f_*([\tilde{\alpha}]) = [\tilde{f} \circ \tilde{\alpha}]$. In case both X and Y are manifolds, it is clear that this homomorphism coincides with the usual homomorphism $\bar{f}_* : \pi_1(|X|, x) \to \pi_1(|Y|, y)$ (that is, there is an isomorphisms $\varphi : \pi_1(X, x) \to \pi_1(|X|, x)$ and $\psi : \pi_1(Y, \bar{f}(x)) \to \pi_1(|Y|, \bar{f}(x))$ such that $\psi \circ f_* = \bar{f}_* \circ \varphi$).

From now on, we shall fix a point $\tilde{x}$ of $p^{-1}(x)$ as the initial point of the representative path $\tilde{\alpha}$ of the element of orbifold fundamental groups. Then, $\tilde{\alpha} \sim \tilde{\beta}$ if and only if $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Let $f : X \to Y$ and $g : Y \to Z$ be orbi-maps. Suppose there exists a point $x \in |X| - \Sigma X$ such that $\bar{f}(x) \in |Y| - \Sigma Y$ and $(\bar{g} \circ \bar{f})(x) \in |Z| - \Sigma Z$. The map $\bar{g} \circ \bar{f} : |X| \to |Y|$ together with the map $\tilde{g} \circ \tilde{f} : |\widetilde{X}| \to |\widetilde{Y}|$ is said to be the *composition* of the orbi-maps $f : X \to Y$ and $g : Y \to Z$, denoted by $(g \circ f) : X \to Z$.

2.1. **Proposition.** *Suppose* $f : (X, x) \to (Y, y)$ *and* $g : (Y, y) \to (Z, z)$ *be orbi-maps. Then* $(g \circ f)_* = g_* \circ f_* : \pi_1(X, x) \to \pi_1(Z, z)$.

Proof. Let $p : (\widetilde{X}, \tilde{x}) \to (X, x)$, $q : (\widetilde{Y}, \tilde{y}) \to (Y, y)$ and $r : (\widetilde{Z}, \tilde{z}) \to (Z, z)$ be the universal coverings. Let $\tilde{f}$ and $\tilde{g}$ be the structure maps of f and g, respectively. Take $[\tilde{\alpha}] \in \pi_1(X, x)$. We have only to show that $((\tilde{g} \circ \tilde{f}) \circ \tilde{\alpha})(1) = (\tilde{g} \circ (\tilde{f} \circ \tilde{\alpha}))(1)$. But it is trivial. Q.E.D.

We use the symbol $[\tilde{\alpha}]_A$ to denote the element of $\mathrm{Aut}(\widetilde{X}, p)$ corresponding to $[\tilde{\alpha}] \in \pi_1(X, x)$ under the isomorphism $\Psi_{\tilde{x}} : \pi_1(X, x) \cong \mathrm{Aut}(\widetilde{X}, p)$, where $\tilde{x} \in |\widetilde{X}|$ is a fixed base point. And we use the symbol $(f_*([\tilde{\alpha}]))_A$ to denote the element of $\mathrm{Aut}(\widetilde{Y}, q)$ corresponding to $f_*([\tilde{\alpha}]) \in \pi_1(Y, \bar{f}(x))$ under the isomorphism $\Psi_{\tilde{f}(\tilde{x})} : \pi_1(Y, \bar{f}(x)) \cong \mathrm{Aut}(Y, \tilde{q})$. In the definition of the orbi-map, condition (03) insure that, in the condition (02), for each σ, there is only one τ. Namely, for $[\tilde{\alpha}]_A \in \mathrm{Aut}(\widetilde{X}, p)$, $\tau = (f_*([\tilde{\alpha}]))_A \in \mathrm{Aut}(\widetilde{Y}, q)$. That is, the following proposition holds.

2.2. **Proposition.** $\tilde{f} \circ [\tilde{\alpha}]_A = (f_*([\tilde{\alpha}]))_A \circ \tilde{f}$, $[\tilde{\alpha}] \in \pi_1(X, x)$.

Proof. Note that $[\tilde{\alpha}]_A$ is characterized as the element of $\mathrm{Aut}(\widetilde{X}, p)$ which transforms $\tilde{\alpha}(0) = \tilde{x}$ to $\tilde{\alpha}(1)$ and $(f_*([\tilde{\alpha}]))_A$ is characterized as the element of $\mathrm{Aut}(\widetilde{Y}, q)$ which transforms $\tilde{f}(\tilde{\alpha}(0))$ to $\tilde{f}(\tilde{\alpha}(1))$. By the condition (02), there exists an element $\tau \in \mathrm{Aut}(\widetilde{Y}, q)$ such that $\tilde{f} \circ [\tilde{\alpha}]_A = \tau \circ \tilde{f}$. On the other hand, $\tau(\tilde{f}(\tilde{\alpha}(0))) = (\tau \circ \tilde{f})(\tilde{\alpha}(0)) = (\tilde{f} \circ [\tilde{\alpha}]_A)(\tilde{\alpha}(0)) = \tilde{f}(\tilde{\alpha}(1))$. That is, τ transforms $\tilde{f}(\tilde{\alpha}(0))$ to $\tilde{f}(\tilde{\alpha}(1))$. Hence $\tau = (f_*([\tilde{\alpha}]))_A$. Q.E.D.

Let X be an orbifold and $p : \widetilde{X} \to X$ the universal covering of it. Let G be a group acting smoothly, effectively, and properly discontinuously on $|\widetilde{X} \times I|$, where $I = [0, 1]$, $|\widetilde{X} \times 1/2| = |\widetilde{X}|$. $(\widetilde{X} \times I)/G$ is called an *I-bundle over* X, if G preserves the product structure (so we may assume G preserves $|\widetilde{X} \times 1/2|$) and $G|(|\widetilde{X} \times 1/2|) = \mathrm{Aut}(\widetilde{X}, p)$. Namely, if G preserves $|\widetilde{X} \times 0|$ and $|\widetilde{X} \times 1|$, respectively, it is called a *product I-bundle over* X, denoted by $X \times I$. From the uniqueness of the universal covering, we may assume that $(p \times \mathrm{id}) : |\widetilde{X} \times I| \to |X \times I|$ is the universal covering from $\widetilde{X} \times I$ to $X \times I$. So we may also assume that for a product I-bundle, $G = \mathrm{Aut}(\widetilde{X} \times I, (p \times \mathrm{id})) = \mathrm{Aut}(\widetilde{X}, p) \times \mathrm{id}$. Let X and Y be good orbifolds. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the universal coverings. Two orbi-maps $f, g : X \to Y$ are said to be *orbi-homotopic*, denoted by $f \sim g$, if there exists an orbi-map $F : X \times I \to Y$ such that $F \mid (X \times 0) = f$, $F \mid (X \times 1) = g$. $F \mid (X \times t)$, $t \in |I|$, is sometimes denoted by f_t and an orbi-homotopy $F : X \times I \to Y$ is sometimes denoted by the terminology an orbi-homotopy $f_t : X \to Y$, $t \in |I|$. Note that f_t is not always an orbi-map since $\overline{F} | (|X \times t|)$ does not always satisfies (03) (though $\widetilde{F} | (|\widetilde{X} \times t|)$ and $\overline{F} | (|X \times t|)$ satisfy (01) and (02).) Note also that if f_t is an orbi-map, then for each $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, there is a $\tau \in \mathrm{Aut}(\widetilde{Y}, q)$ such that $\tilde{f}_t \circ \sigma = \tau \circ \tilde{f}_t$ (τ depends only on σ not on t). Namely, $\tilde{f} \circ \sigma = \tau \circ \tilde{f}$ if and only if $\tilde{g} \circ \sigma = \tau \circ \tilde{g}$. It is easy to see that if f_i, g_i $(i = 1, 2)$ be orbi-maps, $f_1 \sim f_2$, $g_1 \sim g_2$, and orbi-maps $f_1 \circ f_2$ and $g_1 \circ g_2$ are definable, then $f_1 \circ f_2 \sim g_1 \circ g_2$.

2.3. **Proposition.** *Let* $f : (X, x) \to (Y, y)$ *and* $g : (X, x') \to (Y, y')$ *be orbi-maps. Let* $p : \widetilde{X} \to X$ *and* $q : \widetilde{Y} \to Y$ *be the universal coverings. Let* $\tilde{x}$ *and* $\tilde{x}'$ *be any points of* $p^{-1}(x)$ *and* $p^{-1}(x')$, *respectively. Let* r *be any path in* $|\widetilde{X}|$ *from* $\tilde{x}$ *to* $\tilde{x}'$. *Let* s *be any path in* $|\widetilde{Y}|$ *from* $\tilde{f}(\tilde{x})$ *to* $\tilde{g}(\tilde{x}')$, *where* $\tilde{f}$ *and* $\tilde{g}$ *are the structure maps of* f *and* g, *respectively. If* $f \sim g$, *then* $g_* \circ u_r = u_s \circ f_* : \pi_1(X, x) \to \pi_1(Y, y')$.

Proof. Take $\tilde{\alpha} \in \Omega(\widetilde{X}, x)$. We have only to show that $s^{-1} \cdot (\tilde{f} \circ \tilde{\alpha}) \cdot \rho(s) \sim (\tilde{g}(r))^{-1} \cdot (\tilde{g} \circ \tilde{\alpha}) \cdot \tilde{g}(\tau(r))$, where τ is the element of $\mathrm{Aut}(\widetilde{X}, p)$ such that $\tau(\tilde{x}) = \tilde{\alpha}(1)$ and ρ is the element of $\mathrm{Aut}(\widetilde{Y}, q)$ such that $\rho(\tilde{f}(\tilde{x})) = \tilde{f}(\tau(\tilde{x}))$. The initial point of those is $\tilde{g}(\tilde{x}')$. The terminal point of $s^{-1} \cdot (\tilde{f} \circ \tilde{\alpha}) \cdot \rho(s)$ and $(\tilde{g}(r))^{-1} \cdot (\tilde{g} \circ \tilde{\alpha}) \cdot \tilde{g}(\tau(r))$ are $\rho(\tilde{g}(\tilde{x}'))$ and $\tilde{g}(\tau(\tilde{x}'))$, respectively. Since f and g are orbi-homotopic, $\rho \circ \tilde{g} = \tilde{g} \circ \tau$. So $\rho(\tilde{g}(\tilde{x}')) = \tilde{g}(\tau(\tilde{x}'))$. Q.E.D.

Let $p : \widetilde{X} \to X$ be any covering. Let $\hat{p} : \widehat{X} \to X$ be the universal covering and τ be any element of $\mathrm{Aut}(\widehat{X}, \hat{p})$. p together with τ becomes an orbi-map. Conversely, the structure map of any orbi-map whose underlying map is p is some $\tau \in \mathrm{Aut}(\widehat{X}, \hat{p})$. Thus, any covering possesses one and only one orbi-map structure. $p_* : \pi_1(\widetilde{X}, \tilde{x}) \to \pi_1(X, x)$ is defined as the homomorphism induced by the orbi-map whose underlying map is p (this orbi-map uniquely exists). $p(A)$ $(A \subset \widetilde{X})$, $p^{-1}(B)$ $(B \subset X)$, $(p \,|\, A)$, etc. are defined similarly. (Note that both symbols $p(A)$ and $p(|A|)$ have validity.) *An* (*resp. a regular, the universal*) *orbi-covering* is an orbi-map whose underlying map is a (resp. a regular, the universal) covering. Namely, *an orbi-isomorphism* is an orbi-map whose underlying map is an isomorphism.

For orbifold coverings, we can derive the same results as those of ordinary covering spaces. We shall list some propositions which we will need later.

2.4. **Proposition.** *Let* $p' : (X', x') \to (X, x)$ *be any covering. The homomorphism* $p'_* : \pi_1(X', x') \to \pi_1(X, x)$ *is monic.*

Proof. Let $p : (\widetilde{X}, \tilde{x}) \to (X', x')$ be the universal covering. Take $[\tilde{\alpha}] \in \pi_1(X', x')$. We may assume that $\tilde{\alpha}$ is a path in $|\widetilde{X}|$ such that $\tilde{\alpha}(0) = \tilde{x}$ and $p(\tilde{\alpha}(1)) = x'$. From the definition of p'_*, $p'_*([\tilde{\alpha}]) = [\mathrm{id} \circ \tilde{\alpha}] = [\tilde{\alpha}]$. Hence, if $p_*([\tilde{\alpha}]) = 1$ in $\pi_1(X, x)$, then $\tilde{\alpha}(0) = \tilde{\alpha}(1)$. This means that $[\tilde{\alpha}] = 1$ in $\pi_1(X', x')$. Q.E.D.

2.5. **Proposition.** *For any subgroup G of $\pi_1(X, x)$, there exists a covering $p' : (X', x') \to (X, x)$ such that $p'_*\pi_1(X', x') = G$.*

Proof. Let $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ be the universal covering. Regarding G as a subgroup of $\mathrm{Aut}(\widetilde{X}, p)$, put $X' = \widetilde{X}/G$. Define a map p' from $|\widetilde{X}/G|$ to $|\widetilde{X}/\mathrm{Aut}(\widetilde{X}, p)|$ by $p'(\tilde{z}G) = \tilde{z}\,\mathrm{Aut}(\widetilde{X}, p)$, for $\tilde{z} \in |\widetilde{X}|$. It is easy to show that p' is well-defined and the desired covering. Q.E.D.

2.6. **Proposition.** *If $p' : (X', x') \to (X, x)$ is any covering, then for any $x_0 \in |X| - \Sigma X$, $\#p'^{-1}(x_0) = |\pi_1(X, x); p'_*\pi_1(X', x')|$.*

Proof. Let $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ be the universal covering. There exists a covering $q : (\widetilde{X}, \tilde{x}) \to (X', x')$ such that $p = p' \circ q$. Put $G = p_*\pi_1(X', x')$ and $A = \pi_1(X, x)$. Take any point $\tilde{x}_0 \in p^{-1}(x_0)$. Since $p^{-1}(x_0) = \bigcup_{a \in A} a\tilde{x}_0$ and $p'^{-1}(x_0) = q(p^{-1}(x_0))$, $a\tilde{x}_0$ and $ga\tilde{x}_0$ are identified, for any $g \in G$. Hence $\#p'^{-1}(x_0) = |A; G|$. Q.E.D.

Similarly, we can also show that for any suborbifold Y of X and any covering $p : \widetilde{X} \to X$, $\#(p\,|\,\widetilde{Y})^{-1}(y_0) = |\eta_*\pi_1(Y, y); \eta_*(p\,|\,\widetilde{Y})_*\pi_1(\widetilde{Y}, \tilde{y})|$, $y_0 \in |Y| - \Sigma Y$, where $\eta : Y \to X$ is the inclusion orbi-map and $\widetilde{Y}$ is any component of $p^{-1}(Y)$.

From this if $p : (X', x') \to (X, x)$ is a covering and $\pi_1(X', x') \cong \pi_1(X, x)$, then p is an isomorphism.

Let $f : (X, x) \to (Y, y)$ be an orbi-map and $p : (Y', y') \to (Y, y)$ a covering. A *lift* of f with respect to p is an orbi-map $f' : (X, x) \to (Y', y')$ with $\tilde{f}' = \tilde{f}$ and $p \circ \bar{f}' = \bar{f}$.

2.7. **Proposition.** *Let $f : (X, x) \to (Y, y)$ be an orbi-map and $p : (Y', y') \to (Y, y)$ a covering. There exists a lift of f, if and only if $f_*\pi_1(X, x)$ is a subgroup of $p_*\pi_1(Y', y')$.*

Proof. Suppose $f' : (X, x) \to (Y', y')$ is a lift of f. Since $f'_*\pi_1(X, x)$ is a subgroup of $\pi_1(Y', y')$ and $f = p \circ f'$, $f_*\pi_1(X, x) = p_*f'_*\pi_1(X, x) < p_*\pi_1(Y', y')$.

Suppose $f_*\pi_1(X, x) < p_*\pi_1(Y', y')$. Let $q : (\widetilde{X}, \tilde{x}) \to (X, x)$ and $p' : (\widetilde{Y}, \tilde{y}) \to (Y', y')$ be the universal orbi-coverings. Take $\tilde{f} : (\widetilde{X}, \tilde{x}) \to (\widetilde{Y}, \tilde{y})$, the structure map of f. Define a map $\bar{f}'$ from $|X|$ to $|Y'|$ by $\bar{f}'(x) = p'(\tilde{f}(\tilde{x}))$ for $x \in |X|$, where $\tilde{x}$ is any point of $q^{-1}(x)$. It is easy to check that $\bar{f}'$ is well defined and that $f' = (\bar{f}', \tilde{f})$ is the desired orbi-map. Q.E.D.

2.8. **Proposition.** *Let $p : (X', x') \to (X, x)$ be a covering. If $p_*\pi_1(X', x')$ is a normal subgroup of $\pi_1(X, x)$, then $p : (X', x') \to (X, x)$ is a regular covering.*

Proof. Take any $x'_0, x'_1 \in p^{-1}(x)$. Let α be a path from x'_0 to x'_1 in $|X'| -$

$\Sigma X'$. Let $q : \widetilde{X} \to X'$ be the universal covering. Take $\tilde{x}'_0 \in q^{-1}(x'_0)$. Let $\tilde{\alpha}$ be a lift of α beginning at $\tilde{x}'_0$. Put $\tilde{\alpha}(1) = \tilde{x}'_1$. Note that $[\tilde{\alpha}]_A \in \operatorname{Aut}(\widetilde{X}, p \circ q)$. Since $p_*\pi_1(X', x')$ is normal we can define a map $\tilde{p} : |X'| \to |X'|$ by $\tilde{p}(z') = q([\tilde{\alpha}]_A(\tilde{z}))$, where $\tilde{z}$ is any point of $q^{-1}(z')$. It is clear that $\tilde{p}$ together with $[\tilde{\alpha}]_A$ becomes an orbi-map from X' to X'. It is clear that $p \circ \tilde{p} = p$, $\tilde{p}(x'_0) = x'_1$ and $\tilde{p}$ is onto. Moreover, since $[\tilde{\alpha}]p_*\pi_1(X', x') = p_*\pi_1(X', x')[\tilde{\alpha}]$, $\tilde{p}$ is injective. We can also define the inverse map of $\tilde{p}$ using the inverse path of $\tilde{\alpha}$. Hence the map $\tilde{p}$ is a homeomorphism. Since $[\tilde{\alpha}]_A$ is a diffeomorphism, $\tilde{p}$ is an isomorphism from X' to X'. Thus we can construct an element of $\operatorname{Aut}(X', p)$ taking x'_0 to x'_1. Q.E.D.

2.9. **Proposition.** *If $p : (X', x') \to (X, x)$ is a regular covering, then*
(1) *$p_*\pi_1(X', x')$ is a normal subgroup of $\pi_1(X, x)$ and*
(2) *$\pi_1(X, x)/p_*\pi_1(X', x') \cong \operatorname{Aut}(X', p)$.*

Proof. Let $q : (\widetilde{X}, \tilde{x}) \to (X', x')$ be the universal covering. Since p is regular, we can define a map $\Psi : \pi_1(X, x) \to \operatorname{Aut}(X', p)$ by $\Psi[\tilde{\alpha}] = \tau$, where τ is the element of $\operatorname{Aut}(X', p)$ taking x' to $q(\tilde{\alpha}(1))$. It is easy to check that Ψ is an epimorphism and the kernel is $p_*\pi_1(X', x')$. Q.E.D.

3. Some tools for finitely uniformizable 3-orbifolds

Let X be an orbifold. Let $\alpha : |[0, 1]| \to |X| - \Sigma X$ be a loop in $|X| - \Sigma X$ based at x. Let $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ the universal covering. Since α lies in $|X| - \Sigma X$, there is a unique lift $\tilde{\alpha}$ of α in $|\widetilde{X}|$ with $\tilde{\alpha}(0) = \tilde{x}$. By the symbol $[\tilde{\alpha}]$, we shall mean the element of $\pi_1(X, x)$ represented by such $\tilde{\alpha}$. Let α' be a loop in $|X| - \Sigma X$ based at $x' \neq x$. Take a path l in $|X| - \Sigma X$ from x to x'. Since $l \cdot \alpha' \cdot l^{-1}$ is a loop in $|X| - \Sigma X$ based at x, $[l \cdot \alpha' \cdot l^{-1}]$ has the meaning as an element of $\pi_1(X, x)$ defined above. By the symbol $[\alpha']$, we shall mean the element $[l \cdot \alpha' \cdot l^{-1}]$ of $\pi_1(X, x)$ defined above. If we take another path l' in $|X| - \Sigma X$ from x to x', then $[l \cdot \alpha' \cdot l^{-1}]$ and $[l' \cdot \alpha' \cdot l'^{-1}]$ are conjugate in $\pi_1(X, x)$. If f is a map from usually oriented $|S^1|$ to $|X| - \Sigma X$, we can define $[f] \in \pi_1(X, x)$ in the similar way. If f is an embedding, we call not only f but also the (usually oriented) image $C = f(|S^1|)$ a simple closed curve in $|X| - \Sigma X$. By the symbol $[C]$, we shall mean $[f]$.

A map $f : |S^1| \to |X| - \Sigma X$ is said to be *extendable to an orbi-map from $D^2(n)$ to X* if there exists an orbi-map $F : D^2(n) \to X$ with $\overline{F} \mid |\partial D^2(n)| = f$. Since $f(|S^1|) \subset |X| - \Sigma X$, there is a unique orbi-map with the underlying map f. Hence, a map $f : |S^1| \to |X| - \Sigma X$ is extendable to an orbi-map from $D^2(n)$ to X if and only if there exists an orbi-map $F : D^2(n) \to X$ such that $F \mid \partial D^2(n)$ is equal to the orbi-map whose underlying map is f.

3.1. **Lemma.** *Suppose X be an orbifold equipped with a base point $x \in |X| - \Sigma X$ and $f : |S^1| \to |X| - \Sigma X$ be a map. Then, f is extendable to an orbi-map from D^2 to X, if and only if $[f] = 1$ in $\pi_1(X, x)$.*

Proof. Let $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ be the universal covering and l be a path in $|X| - \Sigma X$ from x to the base point of f. Suppose $F : D^2 \to X$ is an orbi-map with $\overline{F} \mid (|\partial D^2|) = f$. Let $\widetilde{F}$ be the structure map of F. Since the universal covering from D^2 to D^2 is the identity, $\widetilde{F} \mid (|\partial D^2|)$ is a lift of the map f. So the lift of $l \cdot f \cdot l^{-1}$ is a closed curve in $|\widetilde{X}|$. This implies that $[f] = 1$ in $\pi_1(X, x)$.

Suppose $[f] = 1$ in $\pi_1(X, x)$. From the hypothesis, the lift of $l \cdot f \cdot l^{-1}$ is a closed curve in $|\widetilde{X}| - p^{-1}(\Sigma X)$. So there is a lift $f' : |S^1| \to |\widetilde{X}| - p^{-1}(\Sigma X)$ of f. Since $|\widetilde{X}|$ is simply connected, f' is extendable to a map $\widetilde{F}$ from $|D^2|$ to $|\widetilde{X}|$. Since id : $|D^2| \to |D^2|$ is the universal covering from D^2 to D^2, $(p \circ \widetilde{F}, \widetilde{F})$ is the desired orbi-map. Q.E.D.

Let X be an orbifold with a base point $x \in |X| - \Sigma X$ and Y a suborbifold of X with a base point $y \in |Y| - \Sigma X$. Let $\bar{i} : |Y| \to |X|$ be the natural inclusion. Let $p : \widetilde{X} \to X$ be the universal covering and $\widetilde{Y}$ a component of $p^{-1}(Y)$. Let $\tilde{i} : |\widetilde{Y}| \to |\widetilde{X}|$ be the natural inclusion. Let $q : \widehat{Y} \to \widetilde{Y}$ be the universal covering. Define $\hat{i} = \tilde{i} \circ q$, $\bar{i}$ together with $\hat{i}$ becomes an orbi-map. The orbi-map $(\bar{i}, \hat{i})$ is called the *inclusion* orbi-map from Y to X. Let $i : Y \to X$ be the inclusion orbi-map. When $y = x$, the induced homomorphism $i_* : \pi_1(Y, x) \to \pi_1(X, x)$ is obvious. Even if $y \neq x$, by taking a path in $|X| - \Sigma X$ from y to x, we can define the induced homomorphism $i_* : \pi_1(Y, y) \to \pi_1(X, x)$ up to conjugation in $\pi_1(X, x)$. Frequently, by denoting simply $\pi_1(Y, y) \to \pi_1(X, x)$, we shall mean the homomorphism induced by the inclusion. Let $p' : X' \to X$ be any covering. Let Y' be any component of $p'^{-1}(Y)$ with $p'(y') = y$, $y \in Y - \Sigma Y$. Let $i' : Y' \to X'$ be the inclusion orbi-map. Then we may assume that the structure maps of i' and i are equal. Hence, it is easy to see that $p' \circ i' = i \circ (p' \mid y')$. Thus, by 2.1, we have $p'_* \circ i'_* = i_* \circ (p' \mid Y')_* : \pi_1(Y', y') \to \pi_1(Y, y)$.

3.2. **Lemma.** *Let X be a k-orbifold and Y a properly embedded, 2-sided $(k-1)$-suborbifold of X such that* $\mathrm{Ker}(i_* : \pi_1(Y, y) \to \pi_1(X, x)) = 1$. *If $f : |S^1| \to |Y| - \Sigma X$ is a map which is extendable to an orbi-map from D^2 to X then f is extendable to an orbi-map from D^2 to Y.*

Proof. Let $p : (\widetilde{X}, \tilde{x}) \to (X, x)$ be the universal covering. Let l_1 be a path in $|X| - \Sigma X$ from x to y and l_2 be a path in $|Y| - \Sigma X$ from y to the base point of f. From the hypothesis and Lemma 3.1, the lift of $l_1 \cdot l_2 \cdot f \cdot l_2^{-1} \cdot l_1^{-1}$ is a closed curve in $|\widetilde{X}|$. So $l_2 \cdot f \cdot l_2^{-1}$ is a closed curve in $|\widetilde{Y}|$, where $\widetilde{Y}$ is any component of $p^{-1}(Y)$. Hence $[f] = 1$ in $\pi_1(Y, y)$. By Lemma 3.1, f is extendable to an orbi-map from D^2 to Y. Q.E.D.

A *regular neighborhood* of the suborbifold Y in X, denoted by $\mathscr{N}(Y)$, is the quotient of an $\mathrm{Aut}(\widetilde{X}, p)$-equivariant regular neighborhood of each component of $p^{-1}(Y)$ in $\widetilde{X}$, where $p : \widetilde{X} \to X$ is the universal covering. It is easy to see that if Y is a properly embedded, 2-sided, 2-suborbifold in a 3-orbifold X, then $\mathcal{N}(Y)$ is $Y \times I$, if Y is a suborbifold of X consists of a vertex of ΣX, then $\mathcal{N}(Y)$ is a ballic orbifold containing Y, if Y is a ballic orbifold, then $\mathcal{N}(Y)$ is a ballic orbifold containing Y, and so on.

Let $f : |S^1| \to |X| - \Sigma X$ be a map and l a path in $|X| - \Sigma X$ from x to the base point of f. Define $[f]^n = [l \cdot f \cdot l^{-1}]^n \in \pi_1(X, x)$.

3.3. **Lemma.** *Let X be an orbifold and $p : \widetilde{X} \to X$ be the universal covering. If $f : |S^1| \to |X| - \Sigma X$ is a map such that $[f]^n = 1$ in $\pi_1(X, x)$ and that $\langle [f]_A \rangle < \mathrm{Aut}(\widetilde{X}; p)$ has a fixed point in $|\widetilde{X}|$, then f is extendable to an orbi-map from $D^2(n)$ to X.*

Proof. Let $q' : |S^1| \to |S^1|$ be the map defined by $q'(e^{ir}) = e^{inr}$. From the hypothesis that $[f]^n = 1$ in $\pi_1(X, x)$, there is a map $\tilde{f} : |S^1| \to |\widetilde{X}| - p^{-1}(\Sigma X)$ with $p \circ \tilde{f} = f \circ q'$. Denote $|D^2(n)| = c * |S^1|$ and $|D^2| = \tilde{c} * |S^1|$. We may regard the map $q((1-t)\tilde{c} + te^{ir}) = (1-t)c + te^{inr} : |D^2| \to |D^2(n)|$ as the universal covering from D^2 to $D^2(n)$. Note that $\mathrm{Aut}(D^2, q) = \mathrm{Aut}(S^1, q')$. Let l be the arc $\{(1-t)\tilde{c} + t \mid 0 \le t \le 1\} \subset |D^2|$. We shall define a continuous map $F : |D^2| \to |\widetilde{X}|$ as follows: Let $\tilde{x} \in |\widetilde{X}|$ be a fixed point of $\langle [f]_A \rangle$. Define $F \mid l$ by an arc in $|\widetilde{X}|$ from $\tilde{x}$ to $\tilde{f}(1)$. Let σ_A be the element of $\mathrm{Aut}(D^2, q)$ corresponding to a local normal loop in $D^2(n)$ around c. By a similar argument in the proof of 2.2, we have $\tilde{f} \circ \sigma_A = [f]_A \circ \tilde{f}$. (Note that $f(|S^1|) \subset |X| - \Sigma X$.) We can define $F \mid \sigma_A^k l$ by $[f]_A^k \circ (F \mid l) \circ \sigma_A^{-k}$. Let e be the minor sector in $|D^2|$ bounded by l and $\sigma_A l$. Define $F \mid e$ by an extension of $F \mid \partial e$. (Such an extension always exists, since $|\widetilde{X}|$ is simply connected.) We can define $F \mid \sigma_A^k e$ by $[f]_A^k \circ (F \mid e) \circ \sigma_A^{-k}$. By piecing together $(F \mid \sigma_A^k e)$'s, we can define a continuous map $F : |D^2| \to |\widetilde{X}|$. From the construction, F satisfies (02) and (03). Hence we can define a continuous map $\overline{F} : |D^2(n)| \to |X|$ by $\overline{F}(x) = q(F(\tilde{x}))$, where $x \in |D^2(n)|$ and $\tilde{x}$ is any point of $q^{-1}(x)$. It is clear that $(\overline{F}, F)$ is the desired orbi-map. Q.E.D.

An orbifold X is said to be *finitely uniformizable*, if there exists a uniformization $p : X' \to X$ such that $\mathrm{Aut}(X', p)$ is finite. An orbifold is finitely uniformizable if and only if its fundamental group contains a torsion free normal subgroup of finite index.

The equivariant Dehn's Lemma, Loop Theorem and Sphere Theorem of [M-Y 1, 2, 3] show the following results.

3.4. **Theorem** (Dehn's Lemma of Orbifold). *Let M be a finitely uniformizable, compact, and orientable 3-orbifold and C a simple closed curve in $|\partial M| - \Sigma M$. If* [C] *has order n in $\pi_1(M)$ then there exists a discal orbifold $D^2(n)$ properly embedded in M such that $|\partial D^2(n)| = C$.*

Proof. Let $\hat{p} : \widehat{M} \to M$ be the universal covering and $\tilde{p} : \widetilde{M} \to M$ a finite uniformization. There is a covering $q : \widehat{M} \to \widetilde{M}$ such that $\hat{p} = \tilde{p} \circ q$. Let

K be a component of $\tilde{p}^{-1}(C)$ and $\widehat{K}$ a component of $q^{-1}(K)$. From the hypothesis, $\widehat{K}$ is a simple closed curve in $|\widehat{M}|$. So K is also a simple closed curve in $|\widetilde{M}|$. Suppose $(q\,|\,\widehat{K}) : \widehat{K} \to K$ and $(\tilde{p}\,|\,K) : K \to C$ are m- and $\tilde{n}$-sheeted coverings, respectively. Note that $n = m\tilde{n}$.

Case 1. If $n = \tilde{n}$, then $m = 1$. That is, $(q\,|\,\widehat{K}) : \widehat{K} \to K$ is a homeomorphism. So $[K] = 1$ in $\tilde{\pi}_1(\widetilde{M}, \tilde{x})$. Since $|\widetilde{M}|$ is compact and $\mathrm{Aut}(\widetilde{M}, \tilde{p})$ is finite, by Theorem 5 of [M-Y 2], there exists a disc $\widetilde{D}^2$ in $\widetilde{M}$ such that $|\partial\widetilde{D}^2| = K$ and either $g(|\widetilde{D}^2|) \cap |\widetilde{D}^2| = \varnothing$ or $g(|\widetilde{D}^2|) = |\widetilde{D}^2|$ for $g \in \mathrm{Aut}(\widetilde{M}, \tilde{p})$. Hence $\tilde{p}(\widetilde{D}^2) = D^2(n)$ and $|\partial D^2(n)| = C$.

Case 2. If $n \gneqq \tilde{n}$, then $[K]$ has order $m \neq 1$ in $\pi_1(\widetilde{M})$. So $[K]_A \neq 1 \in \mathrm{Aut}(\widehat{M}, q)$. Since $[K]_A \in \mathrm{Aut}(\widehat{M}, q)$ leaves invariant $\widehat{K}$ and $\widehat{K}$ is homotopically trivial in $|\widehat{M}|$, by the Corollary of Theorem 5 of [M-Y 2], $[K]_A$ has a fixed point in $|\widehat{M}|$. On the other hand, $q : \widehat{M} \to \widetilde{M}$ is a usual covering. So each element of $\mathrm{Aut}(\widehat{M}, q)$ except the identity has no fixed point. Contradiction. Q.E.D.

3.5. **Theorem** (Loop Theorem of Orbifold). *Let M be a finitely uniformizable, compact and orientable 3-orbifold and F a compact and connected 2-suborbifold in ∂M. If $\mathrm{Ker}(\pi_1(F) \to \pi_1(M)) \neq 1$ then there exists a discal 2-suborbifold $D^2(n)$ such that $\mathrm{Int}(D^2(n)) \subset \mathrm{Int}(M)$, $\partial D^2(n) \subset F$ and $\partial D^2(n)$ does not bound any discal orbifolds in F.*

Proof. Let $p : \widehat{M} \to M$ be the universal covering and $\widetilde{F}$ any component of $p^{-1}(F)$. Note that $(p\,|\,\widetilde{F}) : \widetilde{F} \to F$ is a covering. Let $q : \widehat{F} \to \widetilde{F}$ be the universal covering. Let $i : F \to M$ be the inclusion orbi-map, and $\hat{\imath} : |\widehat{F}| \to |\widehat{M}|$ the structure map of i. From the hypothesis that $\mathrm{Ker}(i_*) \neq 1$, there exists a path $\tilde{l}$ in $|\widehat{F}| - (p\circ q)^{-1}(\Sigma F)$ such that $\tilde{l}(0) \neq \tilde{l}(1)$, $(p\circ q)(\tilde{l}(0)) = (p\circ q)(\tilde{l}(1))$, and $\hat{\imath}(\tilde{l}(0)) = \hat{\imath}(\tilde{l}(1))$. Let $\tilde{p} : (\widetilde{M}, \tilde{x}) \to (M, x)$ be a finite uniformization and F' a component of $\tilde{p}^{-1}(F)$. There is a covering $\hat{p} : \widehat{M} \to \widetilde{M}$ such that $p = \tilde{p} \circ \hat{p}$. We may assume that $\hat{p}(\widetilde{F}) = F'$. Hence $(\hat{p}\,|\,\widetilde{F}) : \widetilde{F} \to F'$ and $(\tilde{p}|F') : F' \to F$ are also coverings. Let $i' : |F'| \to |\widetilde{M}|$ be the natural inclusion. Put $l = (\hat{p}\,|\,\widetilde{F}) \circ q \circ \tilde{l}$. Note that l is a loop in $|F'|$, $[l] \neq 1$ in $\pi_1(|F'|)$ and, since the closed curve $\hat{\imath} \circ \tilde{l}$ is a lift of $i' \circ l$ under the covering $\hat{p} : \widehat{M} \to \widetilde{M}$, $[i' \circ l] = 1$ in $\pi_1(|\widetilde{M}|)$. Hence, $\mathrm{Ker}(i'_* : \pi_1(|F'|) \to \pi_1(|\widetilde{M}|)) \neq 1$. As in the proof of Theorem 6 of [M-Y 2], we can take a simple closed curve $\widetilde{C}$ in $|F'| - \tilde{p}^{-1}(\Sigma F)$ which belongs to the generators of $\mathrm{Ker}(i'_*)$ and is equivariant under the action of $\mathrm{Aut}(\widetilde{M}, \tilde{p})$. So $p(\widetilde{C}) = C$ is a simple closed curve in $|F| - \Sigma F$ with finite order in $\pi_1(M)$ and does not bound the underlying space of any discal orbifold in F. By 3.4, we can derive the conclusion. Q.E.D.

An orbi-map $f : Y \to X$ is called an *orbi-embedding*, if $f(Y)$ is a suborbifold of X and $f : Y \to f(Y)$ is an orbi-isomorphism . Note that if $\tilde{f}_1$ and $\tilde{f}_2$ are two structure maps of orbi-embeddings $f_1 : Y \to X$ and $f_2 : Y \to X$

with $\bar{f}_1 = \bar{f}_2$, respectively, then there is an element $\tau \in \mathrm{Aut}(\widetilde{X}, p)$ such that $\tau \circ \tilde{f}_1 = \tilde{f}_2$, where $p : \widetilde{X} \to X$ is the universal covering.

Let X be an orbifold and $p : \widetilde{X} \to X$ the universal covering. We shall define $\pi_2(X)$ by $\pi_2(|\widetilde{X}|)$. Let S be a spherical orbifold. Let $f : S \to X$ be an orbi-embedding and $\tilde{f} : |S^2| \to |\widetilde{X}|$ the structure map of f. We use the symbol $[f]$ to denote the element of $\pi_2(|\widetilde{X}|)$ defined by $[\tilde{f}] \in \pi_2(|\widetilde{X}|)$.

3.6. **Theorem** (Sphere Theorem of Orbifold). *Let M be a finitely uniformizable, compact and orientable 3-orbifold. If $\pi_2(M) \neq 0$, then there exist a spherical orbifold S and an orbi-embedding $f : S \to M$ such that $[f] \neq 0$ in $\pi_2(M)$.*

Proof. Let $p : \widetilde{M} \to M$ be a finite uniformization and $\hat{p} : \widehat{M} \to M$ the universal covering. From the hypothesis, $\pi_2(|\widehat{M}|) \neq 0$, so $\pi_2(|\widetilde{M}|) \neq 0$. By Theorem 7 of [M-Y 1] and Lemma 4 of [M-Y 3], there exists a 2-sphere $S^2 \subset \widetilde{M}$ such that $[|S^2|] \neq 0$ in $\pi_2(|\widetilde{M}|)$ and for $g \in \mathrm{Aut}(\widetilde{M}, p)$, $g(|S^2|) \cap |S^2| = \varnothing$ or $g(|S^2|) = |S^2|$. We obtain the desired orbi-embedding by $p(S^2)$. Q.E.D.

A 3-orbifold M is called *irreducible* if any spherical suborbifold in M bounds a ballic suborbifold in M.

3.7. **Lemma.** *If M is a finitely uniformizable, irreducible, compact and orientable 3-orbifold, then $\pi_2(M) = 0$.*

Proof. Suppose $\pi_2(M) \neq 0$. By 3.6, there exist a spherical orbifold S and an orbi-embedding $f : S \to M$ such that $[f] \neq 0$ in $\pi_2(M)$. Let $p : \widehat{M} \to M$ be the universal covering and $\hat{f}$ the structure map of f. From the definition of $\pi_2(M)$, $[\hat{f}] \neq 0$ in $\pi_2(|\hat{M}|)$. On the other hand, from the hypothesis, $f(S)$ bounds a ballic orbifold in M. Hence $f : S \to M$ is extendable to an orbi-map from a ballic orbifold to M. This implies $[\hat{f}] = 0$ in $\pi_2(|\widehat{M}|)$. Contradiction. Q.E.D.

Let M be a 3-orbifold and F a connected 2-suborbifold which is either properly embedded in M or contained in ∂M. We say that F is *incompressible* in M if none of the following conditions is satisfied.

(i) F is a spherical orbifold which bounds a ballic orbifold in M, or

(ii) F is a discal orbifold and either $F \subset \partial M$ or there is a ballic orbifold $X = F \times I \subset M$ with $\partial X \subset F \cup \partial M$, or

(iii) there is a discal orbifold $D^2(n) \subset M$ with $D^2(n) \cap F = \partial D^2(n)$ and $\partial D^2(n)$ does not bound any discal orbifolds in F.

3.8. **Lemma.** *Let M be a finitely uniformizable, compact, and orientable 3-orbifold and F $(\neq S^2(2, 3, 5))$ a compact, 2-sided, and incompressible 2-suborbifold in M. For any covering $p' : M' \to M$, any component of $p'^{-1}(F)$ is incompressible in M'.*

Proof. In case F is a discal or spherical orbifold, we can easily show the conclusion by using Theorem 3 of [M-Y 4]. Otherwise let $M_1 = \mathrm{cl}(M - F \times [0, 1])$.

Note that in case $F \subset \partial M$, M_1 is orbi-isomorphic to M. Since F is incompressible in M, $F \times 0$ is incompressible in M_1. It is easy to see that if each component of $p'^{-1}(F \times i)$, $i = 0, 1$, is incompressible in the component of $p'^{-1}(M_1)$ which includes the component of $p'^{-1}(F \times i)$, then each component of $p'^{-1}(F)$ is incompressible in M'. So we suppose a component $F' \times 0$ of $p'^{-1}(F \times i)$ is not incompressible in the component M_1' of $p'^{-1}(M_1)$ which includes $F' \times 0$. So we may assume that there is a discal orbifold $D^2(n) \subset M_1'$ with $D^2(n) \cap (F' \times 0) = \partial D^2(n)$ and $\partial D^2(n)$ does not bound any discal orbifolds in $F' \times 0$. Since $(p' \mid F' \times 0)_* : \pi_1(F' \times 0) \to \pi_1(F \times 0)$ is monic, this implies that $\operatorname{Ker}(\pi_1(F \times 0) \to \pi_1(M_1)) \neq 1$. Hence, by 3.5, $F \times 0$ is not incompressible in M_1. Contradiction. Q.E.D.

Let M be a 3-orbifold and F a 2-suborbifold properly embedded in M. A 3-orbifold M' (not necessarily connected) is said to be *obtained by cutting M open along F*, if $M' = \operatorname{cl}(M - F \times I)$.

3.9. **Lemma.** *Let M be a finitely uniformizable, compact and orientable 3-orbifold and F a compact, 2-sided, and incompressible 2-suborbifold in M. Suppose M' is obtained by cutting M open along F. Then, for each component N of M', the following* (a) *and* (b) *hold;*

(a) *If M is irreducible, then N is irreducible.*

(b) $\operatorname{Ker}(\pi_1(N) \to \pi_1(M)) = 1$.

Proof. (a) Let S be a spherical orbifold in N. From the irreducibility of M, there is a ballic orbifold $B \subset M$ such that $\partial B = S$. Since F is incompressible in M, $|B| \cap |F| = \emptyset$, so we have $B \subset N$.

(b) Let $p : \widetilde{M} \to M$ be a finite uniformization and $\hat{p} : \widehat{M} \to \widetilde{M}$ the universal covering. Let $\widetilde{N}$ be a component of $p^{-1}(N)$ and $q : \widehat{N} \to \widetilde{N}$ the universal covering. Since $(p \mid \widetilde{N}) : \widetilde{N} \to N$ is a covering, $(p \mid \widetilde{N}) \circ q : \widehat{N} \to N$ is the universal covering. Take $[\alpha] \in \pi_1(N, x')$, $x' \in |N| - \Sigma N$, where α is a loop in $|N| - \Sigma N$. Let $\tilde{\alpha}$ be a lift of α in $|\widehat{N}|$, $i : N \to M$ the inclusion orbi-map and $\tilde{i} : |\widetilde{N}| \to |\widetilde{M}|$ the natural inclusions. Let $\hat{i}$ be the structure map of i (automatically $\hat{p} \circ \hat{i} = \tilde{i} \circ q$.) If $i_*([\alpha]) = [\hat{i} \circ \tilde{\alpha}] = 1$ in $\pi_1(M, x')$, then $\hat{i} \circ \tilde{\alpha}$ is a closed curve in $|\widehat{M}|$. Then $\hat{p} \circ \hat{i} \circ \tilde{\alpha} = \tilde{i} \circ q \circ \tilde{\alpha}$ is a closed curve in $|\widetilde{M}|$. Hence $q \circ \tilde{\alpha}$ is a closed curve in $|\widetilde{N}|$ and the closed curve $\hat{i} \circ \tilde{\alpha}$ is a lift of $\tilde{i} \circ q \circ \tilde{\alpha}$ under the covering $\hat{p} : \widehat{M} \to \widetilde{M}$. Thus $\tilde{i}_*([q \circ \tilde{\alpha}]) = 1$ in $\pi_1(|\widetilde{M}|)$. By 3.8, since $\operatorname{Ker}(\tilde{i}_* : \pi_1(|\widetilde{N}|) \to \pi_1(|\widetilde{M}|)) = 1$, $[q \circ \tilde{\alpha}] = 1$ in $\pi_1(|\widetilde{N}|)$. Hence $\tilde{\alpha}$ is a closed curve in $|\widehat{N}|$. Q.E.D.

3.10. **Lemma.** *Let M be a finitely uniformizable, compact and orientable 3-orbifold and F a compact, 2-sided, and incompressible 2-suborbifold in M. Then* $\operatorname{Ker}(i_* : \pi_1(F) \to \pi_1(M)) = 1$.

Proof. Let N be a component of $\operatorname{cl}(M - F \times [0, 1])$ containing $F \times 0$. From the incompressibility of F and 3.5, $\operatorname{Ker}(\pi_1(F \times 0) \to \pi_1(N)) = 1$. By 3.9, $\operatorname{Ker}(\pi_1(N) \to \pi_1(M)) = 1$. Hence $\operatorname{Ker}(\pi_1(F \times 0) \to \pi_1(M)) = 1$. Q.E.D.

4. A CONSTRUCTION OF AN ORBI-MAP

Let S be a spherical orbifold and B be the ballic orbifold which is the cone on S. By denoting $B = c * S$, we shall mean that B is the cone on S and c is the cone point of ΣB.

4.1. **Lemma.** *Let M be an orientable 3-orbifold such that the underlying space of the universal covering orbifold of* $\mathrm{Int}(M)$ *is homeomorphic to* $\mathbf{R}^3$. *Let S be any spherical orbifold. Then, for any orbi-map $f : S \to \mathrm{Int}(M)$, there exists an orbi-map g from the cone on S to M such that $g\,|\,S = f$.*

Proof. Let $p : S^2 \to S$ and $q : \widetilde{M} \to M$ be the universal coverings. Let $\tilde{f}$ be the structure map of f. Let e and e' be 2-discs on the underlying space of S such that $|S| = e \cup e'$, $\partial e = \partial e'$ and $\Sigma S \subset \partial e$. Let $\tilde{e}$ be the closure of a connected component of $p^{-1}(e) - p^{-1}(\Sigma S)$ and $\tilde{e}'$ be the closure of a connected component of $p^{-1}(e') - p^{-1}(\Sigma S)$. (See Figure 4.1.) We can describe

$$|S^2| = \bigcup_{\sigma_A \in \mathrm{Aut}(S^2, p)} (\sigma_A \tilde{e} \cup \sigma_A \tilde{e}').$$

Let B be the cone on S. We shall denote $B = c * S$. Let $\bar{p} : B^3 \to B$ be the universal covering and $\tilde{c} = \bar{p}^{-1}(c)$. We can describe

$$|B^3| = \bigcup_{\sigma_A \in \mathrm{Aut}(S^2, p)} (\tilde{c} * (\sigma_A \tilde{e}) \cup \tilde{c} * (\sigma_A \tilde{e}')).$$

We may assume $\bar{p}((1 - t)\tilde{c} + t\tilde{x}) = (1 - t)c + tp(\tilde{x})$, where $\tilde{x} \in |S^2|$. Note that $\mathrm{Aut}(S^2, p) = \mathrm{Aut}(B^3, \bar{p})$. From the hypothesis, we can regard $(f_* \pi_1(S))_A$ as a finite subgroup of $\mathrm{diff}_+(\mathbf{R}^3)$. Hence, by Corollary I.1b of [B-K-S] (also by Theorem 4 of [M-Y 4] in case $(f_* \pi_1(S))_A \not\cong A_5$), there exists a point $\tilde{d}$ in $\mathrm{Int}(|\widetilde{M}|)$ fixed by $(f_* \pi_1(S))_A$.

For any fixed point $\tilde{y} \in \mathrm{Int}(|\widetilde{M}|)$ of an element $\tau_A \in (f_* \pi_1(S))_A$, there is a line segment in $\mathrm{Int}(|\widetilde{M}|)$ from $\tilde{d}$ to $\tilde{y}$ fixed by τ_A. Note that, if $\tilde{x} \in |S^2|$ is fixed by $\sigma_A \in \mathrm{Aut}(S^2, p)$, then $\tilde{f}(\tilde{x})$ is fixed by $f_*(\sigma)_A$, since $f_*(\sigma)_A(\tilde{f}(\tilde{x})) = \tilde{f}(\sigma_A(\tilde{x})) = \tilde{f}(\tilde{x})$, by 2.2.

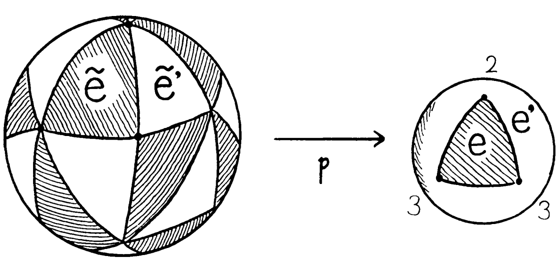


FIGURE 4.1

Let $Q = c * e$, $Q' = c * e'$, and $\{x_1, x_2, x_3\} = \Sigma S$. Let l_i be the arc $c * x_i$ and L_i the arc in ∂e such that $\partial L_i = \{x_j, x_k\}$ and $x_i \notin L_i$, where $\{i, j, k\} = \{1, 2, 3\}$. Let $E_i = c * L_i$. Let $\tilde{x}_i = p^{-1}(x_i) \cap \tilde{e}$, $\widetilde{Q} = \tilde{c} * \tilde{e}$, $\widetilde{Q}' = \tilde{c} * \tilde{e}'$, $\tilde{l}_i = \bar{p}^{-1}(l_i) \cap \widetilde{Q}$, and $\widetilde{E}_i = \bar{p}^{-1}(E_i) \cap \widetilde{Q}$. We shall define a continuous map $\tilde{g} : |B^3| \to |\widetilde{M}|$ as follows: Let $(\sigma_i)_A$ be the element of $\mathrm{Aut}(S^2, p)$ fixing $\tilde{x}_i$, corresponding to a local normal loop in S around x_i. Define $\tilde{g} \,|\, \tilde{l}_i$ by the above line segment fixed by $(f_*(\sigma_i))_A$. Let ρ_A be any element of $\mathrm{Aut}(B^3, \bar{p})$. We can define $\tilde{g} \,|\, \rho_A \tilde{l}_i$ by $(f_*(\rho))_A \circ (\tilde{g} \,|\, \tilde{l}_i) \circ \rho_A^{-1}$. Define $\tilde{g} \,|\, \widetilde{E}_i$ by an extension of $\tilde{g} \,|\, \partial \widetilde{E}_i$. (Such an extension always exists, since $|\widetilde{M}|$ is simply connected.) We can define $\tilde{g} \,|\, \rho_A \widetilde{E}_i$ by $(f_*(\rho))_A \circ (\tilde{g} \,|\, \widetilde{E}_i) \circ \rho_A^{-1}$. Define $\tilde{g} \,|\, \widetilde{Q}$ and $\tilde{g} \,|\, \widetilde{Q}'$ by extensions of $\tilde{g} \,|\, \partial \widetilde{Q}$ and $\tilde{g} \,|\, \partial \widetilde{Q}'$, respectively. (Such extensions always exist, since $\pi_2(|\widetilde{M}|) = 0$.) We can define $\tilde{g} \,|\, \rho_A \widetilde{Q}$ and $\tilde{g} \,|\, \rho_A \widetilde{Q}'$ by $(f_*(\rho))_A \circ (\tilde{g} \,|\, \widetilde{Q}) \circ \rho_A^{-1}$ and $(f_*(\rho))_A \circ (\tilde{g} \,|\, \widetilde{Q}') \circ \rho_A^{-1}$, respectively. By piecing together $(\tilde{g} \,|\, \rho_A \widetilde{Q})$'s and $(\tilde{g} \,|\, \rho_A \widetilde{Q}')$'s, we can define a continuous map $\tilde{g} : |B^3| \to |\widetilde{M}|$.

From the construction, $\tilde{g}$ satisfies (02) and (03). Hence we can define a map $\bar{g} : |B| \to |M|$ by $\bar{g}(x) = q(\tilde{g}(\tilde{x}))$, where $x \in |B|$ and $\tilde{x}$ is any point of $\bar{p}^{-1}(x)$. It is clear that $(\bar{g}, \tilde{g})$ is the desired orbi-map. Q.E.D.

Let S be a triangulable topological space and (K, t) a simplicial division of S. From now on, we identify S and the underlying polyhedron $|K|$ so that we call the simplicial complex K a simplicial division of S. Let X be an n-orbifold. A simplicial division K_X of $|X|$ is called a *simplicial division of* X if the k-dimensional strata is included in the polyhedron consists of $\bigcup\{e \mid e \in K_X^{(k)}\}$ and for each n-simplex e, we have either $e \cap \Sigma X = \varnothing$ or there is only one proper face e' of e such that $e \cap \Sigma X = e'$. An orbifold which has a simplicial division is said to be *triangulable.* It is easy to see that if K_X is a simplicial division of X, then any subdivision of K_X is also a simplicial division of X. Note that orientable 2- or 3-orbifolds are triangulable. Furthermore, if F is a properly embedded, 2-sided, 2-suborbifold in an orientable 3-orbifold M, then for any triangulation K_F of F, there exists a triangulation K_M of M such that K_F is a subcomplex of K_M. These are proved by an analog of the argument in the proof of 5.2 (using the product structures and mixing triangulations). Let X and Y be triangulable orbifolds. Let K_X and K_Y be the simplicial divisions of X and Y, respectively. An orbi-map $f : X \to Y$ is called a *simplicial orbi-map* with respect to K_X and K_Y, if $\bar{f}$ is a simplicial map from $|X|$ $(= |K_X|)$ to $|Y|$ $(= |K_Y|)$ with respect to K_X and K_Y, where $|K|$ is the underlying polyhedron of the complex K.

Let M be a 2- or 3-orbifold. Let $p : \widetilde{M} \to M$ be any covering. Note that if K_M is any simplicial division of M then the pull back of K_M under p gives a simplicial division of $\widetilde{M}$. We will denote the simplicial division of $\widetilde{M}$ by $\widetilde{K}_M$. With these simplicial divisions, the orbi-covering with the underlying map p is a simplicial orbi-map. Moreover, p is a homeomorphism on each simplex e_A of $\widetilde{K}_M$ to a simplicial e_B of K_M.

Let M be an n-orbifold, $n = 2, 3$. Let $p : (\widetilde{M}, \tilde{x}) \to (M, x)$ be the universal covering. Let K_M be a simplicial division of M. Let K_M^* be the dual cell division of K_M. Let T be a maximal tree of K_M^*. We may assume that T is including x. Let $\widetilde{T}$ be the lift of T under p including $\tilde{x}$. Put $D = \{e \mid e$ is an n-simplex of $\widetilde{K}_M$ with $e \cap |\widetilde{T}| \neq \emptyset\}$. Note that $(p \mid D) : D \to |M|$ is one to one except on $(n-1)$-faces of some e's. And note that $(p \mid D)$ is two to one on interior of such faces. We remove one of all such faces from D and attach $(n-2)$ faces of such faces which are the preimage of ΣM to get a domain $\widetilde{M}_e$ in $|\widetilde{M}|$ such that $(p \mid \widetilde{M}_e) : \widetilde{M}_e \to |M|$ is one to one. We call such $\widetilde{M}_e$ a *fundamental domain* of $p : \widetilde{M} \to M$.

4.2. **Theorem.** *Let M be a compact 2- or 3-orbifold and N an orientable 3-orbifold such that the underlying space of the universal covering orbifold of* $\mathrm{Int}(N)$ *is homeomorphic to* $\mathbf{R}^3$. *For any homomorphism $\varphi : \pi_1(M) \to \pi_1(N)$, there exists an orbi-map $f : M \to N$ such that $f_* = \varphi$.*

Proof. Let $U(\Sigma M)$ be a regular neighborhood (in the ordinary sense) of ΣM in $|M|$. Let K_0 be a simplicial division of $|M| - U(\Sigma M)$. Let $K_0^{(r)}$ be the r-skeleton of K_0. Fix a base point x included in $K_0^{(1)}$. Let T be a maximal tree including x. We can describe $K_0^{(1)} = T \cup \{k_1, \ldots, k_r\}$, where $k_1, \ldots, k_r$ are 1-simplices in $K_0^{(1)}$. Let $p : \widetilde{M} \to M$ be the universal covering. By the symbol $\widetilde{K}_0$, we shall mean the pull back of K_0 under p. Fix a base point $\tilde{x} \in p^{-1}(x)$. Note that we can find one and only one simple closed curve in $|T \cup k_i|$. Let $\tilde{k}_i$ be the lift of the simple closed curve under p with the initial point $\tilde{x}$. Since $\tilde{k}_i \in \Omega(\widetilde{M}, x)$, $[\tilde{k}_i] \in \pi_1(M, x)$. Let $q : \widetilde{N} \to N$ be the universal covering.

Step 1. We construct maps $\tilde{f}_1 : |\widetilde{K}_0^{(1)}| \to |\widetilde{N}| - q^{-1}(\Sigma N)$ and $\bar{f}_1 : |K_0^{(1)}| \to |N| - \Sigma N$ which satisfy (01), (02), (03) and $[\tilde{f}_1 \circ \tilde{k}_i] = \varphi[\tilde{k}_i] \in \pi_1(N)$. (Automatically, τ of (02) is $\varphi[\tilde{k}_i]$.)

Fix a base point y in $N_0 = |N| - \Sigma N$. Define $\bar{f}_1(|T|) = y$. Fix $\tilde{y} \in q^{-1}(y)$. Let $\tilde{l}_i$ be a representative of $\varphi[\tilde{k}_i]$ (it is a path in $|\widetilde{N}| - q^{-1}(\Sigma N)$). Note that $q(\tilde{l}_i) = l_i$ is a loop in N_0. Define $\bar{f}_1(k_i) = l_i$. Thus we get a map $\bar{f}_1 : |K_0^{(1)}| \to N_0$. Let $\eta : \pi_1(|K_0^{(1)}|) \to \pi_1(M)$ and $\xi : \pi_1(N_0) \to \pi_1(N)$ be the obvious epimorphisms. From the construction of $\bar{f}_1$, it holds that $\xi \circ \bar{f}_{1*} = \varphi \circ \eta$. So there exists a lift (in the ordinary sense) $\tilde{f}_1 : |\widetilde{K}_0^{(1)}| \to q^{-1}(N_0)$. It is easy to verify that these $\bar{f}_1$ and $\tilde{f}_1$ are the desired maps.

Let $\widetilde{M}_e$ be a fundamental domain of $p : \widetilde{M} \to M$ including $\tilde{x}$. By the symbol (z, σ), $z \in |M|$, $\sigma \in \pi_1(M, x)$, we shall mean $\sigma_A(\tilde{z}) \in |\widetilde{M}|$, where $\tilde{z}$ is one point in $p^{-1}(z) \cap \widetilde{M}_e$.

Step 2. We construct maps $\tilde{f}_2 : |\widetilde{K}_0(2)| \to |\widetilde{N}|$ and $\bar{f}_2 : |K_0^{(2)}| \to |N|$ which satisfy (01), (02), (03), $\tilde{f}_2 \mid |\widetilde{K}_0^{(1)}| = \tilde{f}_1$ and $\bar{f}_2 \mid |K_0^{(1)}| = \bar{f}_1$.

Let K' be the collection of all $e^{(2)} \in \widetilde{K}_0^{(2)}$ such that any point in $\mathrm{Int}(e^{(2)})$ is

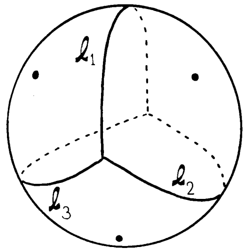


FIGURE 4.2

expressed by $(z, 1)$, $z \in |K_0^{(2)}|$, $1 \in \pi_1(M, x)$. Since $|\widetilde{N}|$ is simply connected, we can extend the map $\tilde{f}_1 | \partial e^{(2)}$ to the map $\rho : e^{(2)} \to |\widetilde{N}|$. Note that, for any $\sigma_A \in \mathrm{Aut}(\widetilde{M}, p)$ and $\sigma_A(e^{(2)})$, $\varphi(\sigma)_A \circ \rho \circ \sigma_A^{-1}$ is a map from $\sigma_A(e^{(2)})$ to $|\widetilde{N}|$ and an extension of $\tilde{f}_1 | \partial(\sigma_A(e^{(2)}))$. By piecing together above extensions on each $e^{(2)} \in K'$ and $\sigma_A(e^{(2)})$, $e^{(2)} \in K'$, we can get a map $\tilde{f}_2 : |\widetilde{K}_0^{(2)}| \to |\widetilde{N}|$. Define $\bar{f}_2 : |K_0^{(2)}| \to |N|$ by $\bar{f}_2(z) = (q \circ \tilde{f}_2)(\tilde{z})$, where $z \in |K_0^{(2)}|$ and $\tilde{z}$ is any point in $p^{-1}(z)$. Such $\tilde{f}_2$ and $\bar{f}_2$ satisfy (01), (02), (03), $\tilde{f}_2 | |\widetilde{K}_0^{(1)}| = \tilde{f}_1$ and $\bar{f}_2 | |K_0^{(1)}| = \bar{f}_1$. In case $\dim M = 2$, by 3.3, the proof is completed.

Step 3. We construct maps $\tilde{f}_3 : |\widetilde{K}_0^{(3)}| \to |\widetilde{N}|$ and $\bar{f}_3 : |K_0^{(3)}| \to |N|$ which satisfy (01), (02), (03), $\tilde{f}_3 | |\widetilde{K}_0^{(2)}| = \tilde{f}_2$ and $\bar{f}_3 | |K_0^{(2)}| = \bar{f}_2$.

This can be done by the same way as Step 2, since $\pi_2(|\widetilde{N}|) = 0$. Let $z_0 \in \Sigma^{(0)} M$. By the symbol $B(z_0)$, we shall mean the regular neighborhood of z_0 in M. Note that $B(z_0)$ is a ballic orbifold. By the symbol $S(z_0)$, we shall mean the spherical orbifold $\partial B(z_0)$. We may assume that $|B(z_0)| \cap |K_0| = l_1 \cup l_2 \cup l_3$, where l_i's are simple arcs in $|S(z_0)|$ such that $l_i \cap l_j = \partial l_i = \partial l_j$ and that each component of $|S(z_0)| - (l_1 \cup l_2 \cup l_3)$ includes one and only one singular point of $S(z_0)$. (See Figure 4.2.) Put $L = |K_0| \cup (\bigcup_{z_0 \in \Sigma^{(0)} M} |S(z_0)|)$ and

$$\widetilde{L} = |\widetilde{K}_0| \cup \left(\bigcup_{z_0 \in \Sigma^{(0)} M} p^{-1}(|S(z_0)|) \right).$$

Step 4. We construct maps $\tilde{f}_4 : \widetilde{L} \to |\widetilde{N}|$ and $\bar{f}_4 : L \to |N|$ which satisfy (01), (02), (03), $\tilde{f}_4 | |\widetilde{K}_0^{(3)}| = \tilde{f}_3$ and $\bar{f}_4 | |K_0^{(3)}| = \bar{f}_3$.

Note that the orbifold whose underlying space is the closure of each component of $|S(z_0)| - (l_1 \cup l_2 \cup l_3)$ is a discal orbifold. Let $D(z_0)$ be one of those discal orbifolds. If necessary, by modifying with an orbi-homotopy, we may assume that $\bar{f}_4(|\partial D(z_0)|) \subset |N| - \Sigma N$. Hence, we can apply 3.3 to show Step 4.

Step 5. Since the remaining parts are the cones on spherical orbifolds, by 4.1, we extend the map and get a desired orbi-map from M to N. Q.E.D.

By almost the same way as 4.1, we can show the following results.

4.3. **Lemma.** *Let F be any orientable and nonspherical 2-orbifold. Let S be any spherical orbifold. Then, for any orbi-map $f : S \to \mathrm{Int}(F)$, there exists an orbi-map $g : c * S \to F$ such that $g \mid S = f \mid S$.*

Proof. Let $q : \widetilde{F} \to F$ be the universal covering. Since F is not spherical, $|\mathrm{Int}(\widetilde{F})|$ is homeomorphic to $\mathbf{R}^2$. Since $\dim \Sigma F = 0$, each finite subgroup G of $\mathrm{Aut}(\widetilde{F}, q)$ has a fixed point in $|\mathrm{Int}(\widetilde{F})|$. Let $\tilde{d}$ be the fixed point of $(f_* \pi_1(S))_A$. We shall use the same symbols as in the proof of 4.1. We can define a map $\tilde{g}_e : |\tilde{c} * \tilde{e}| \to |\widetilde{F}|$ by $\tilde{g}_e((1-t)\tilde{c} + t\tilde{y}) = \tilde{d}$. The remaining parts of the proof are the same as the proof of 4.1. Q.E.D.

4.4. **Proposition.** *Let M be a compact 2- or 3-orbifold and F any orientable and nonspherical 2-orbifold. For any homomorphism $\varphi : \pi_1(M) \to \pi_1(F)$, there exists an orbi-map $f : M \to F$ such that $f_* = \varphi$.*

Proof. By 4.3, we can prove this in the same way as 4.2. Q.E.D.

5. Modifications of orbi-maps

For a simplicial complex K and $x \in K^{(0)}$, the open star neighborhood of x with respect to K, $O(x, K)$, is the interior of the underlying polyhedron of $\mathrm{st}(x, K)$.

5.1. **Theorem.** *Suppose M, N are compact and triangulable 2- or 3-dimensional orbifolds and $f : M \to N$ is an orbi-map. Then, for any simplicial division K_N of N, there exist simplicial divisions K_M of M and a simplicial orbi-map $g : M \to N$ with respect to K_M and K_N such that f and g are orbi-homotopic.*

Proof. (I) Let $\bar{f} : |M| \to |N|$ be the underlying map of f. Since M is compact and $\bar{f}$ is continuous, there exist simplicial divisions K_M of M which satisfy the following condition; for any vertex $v \in K_M^{(0)}$, there exists a vertex $w \in K_N^{(0)}$ such that $O(v, K_M) \subset \bar{f}^{-1}(O(w, K_N))$. Hence $\bar{f}(O(v, K_M)) \subset O(w, K_N)$. Note that such w is not unique with respect to v. For each $v \in K_M^{(0)}$, correspond one of those points, $w(v)$. We may assume that there exists a vertex $x_0 \in K_M^{(0)}$ with $\bar{f}(x_0) \in K_N^{(0)} \cap (|N| - \Sigma N)$. Let $p : \widetilde{M} \to M$ and $q : \widetilde{N} \to N$ be the universal coverings. Let $\tilde{f} : |\widetilde{M}| \to |\widetilde{N}|$ be the structure map of f. It is clear that for each $\tilde{v} \in p^{-1}(v)$, there exists only one $\tilde{w}(\tilde{v}) \in q^{-1}(w(v))$ such that $\tilde{f}(O(\tilde{v}, \widetilde{K}_M)) \subset O(\tilde{w}(\tilde{v}), \widetilde{K}_N)$. (Note that we have already fixed a w for each v.)

(II) Define a map $\tilde{g}_0$ from $\widetilde{K}_M^{(0)}$ to $\widetilde{K}_N^{(0)}$ by $\tilde{g}_0(\tilde{v}) = \tilde{w}(\tilde{v})$. We can easily show that $\tilde{g}_0$ satisfies the condition that $\tilde{g}_0 \circ \sigma_A = (f_*(\sigma))_A \circ \tilde{g}_0$, $\sigma \in \pi_1(M)$. Define a map $\tilde{g}$ from $\widetilde{K}_M$ to $\widetilde{K}_N$ by piecing together the linear extension of $\tilde{g}_0$ on each simplex of $\widetilde{K}_M$. It is easy to check that $\tilde{g}$ also satisfies the condition that $\tilde{g} \circ \sigma_A = (f_*(\sigma))_A \circ \tilde{g}$, $\sigma \in \pi_1(M)$. Thus we can define a map $\bar{g} : |M| \to |N|$ by $\bar{g}(x) = q(\tilde{g}(\tilde{x}))$, $x \in |M|$, where $\tilde{x}$ is any point of $p^{-1}(x)$. Note that $\bar{g}(x_0) = \bar{f}(x_0) \in |N| - \Sigma N$. Since $\tilde{g}$ is simplicial, $\bar{g}$ is also simplicial. It is clear $(\bar{g}, \tilde{g})$ is the desired orbi-map.

(III) We shall define an orbi-map $F : M \times I \to N$ in the following manner: For $(\tilde{x}, t) \in |\widetilde{M \times I}|$, define

$$\widetilde{F}(\tilde{x}, t) = t\tilde{g}(\tilde{x}) + (1-t)\tilde{f}(\tilde{x}).$$

Since for each $\tilde{x} \in |\widetilde{M}|$, both $\tilde{f}(\tilde{x})$ and $\tilde{g}(\tilde{x})$ are included in a same simplex of $\widetilde{K}_N$, $\widetilde{F}$ satisfies (02). Hence, we can define an orbi-map $\overline{F} : |M \times I| \to |N|$ by $\overline{F}(x, t) = q(\widetilde{F}(\tilde{x}, t))$, $x \in |M|$, where $\tilde{x}$ is any point of $p^{-1}(x)$. Note that $\overline{F}(x_0, t) = \bar{f}(x_0) \in |N| - \Sigma N$ for any $t \in |I|$. It is easy to see that $(\overline{F}, \widetilde{F})$ is the desired orbi-homotopy. Q.E.D.

Let X and Y be orbifolds and $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the universal coverings. Note that $(p \times q) : \widetilde{X} \times \widetilde{Y} \to X \times Y$ is the universal covering. Let $f : X \to X'$ and $g : Y \to Y'$ be orbi-maps. *The product of orbi-maps f and g*, denoted by $f \times g$, is the orbi-map whose underlying map and structure map are $\bar{f} \times \bar{g}$ and $\tilde{f} \times \tilde{g}$, respectively.

5.2. *Remark.* In 5.1, suppose that for a component A of ∂M, there is a component B of ∂N such that $f(A) \subset B$ and $(f|A) : A \to B$ is an orbi-covering. Then we can construct an orbi-homotopy F between f and g such that $F|(A \times t) = f$, for $t \in |I|$.

Proof. Since $(f|A) : A \to B$ is an orbi-covering, we may assume that M is triangulated so that $\bar{f}$ is a homeomorphism from each simplex in $|A|$ to a simplex in $|B|$. Let $A \times I$ be the product neighborhood of $A = A \times 0$ in M. Triangulate $\mathrm{cl}(M - (A \times I))$ by mapping the above triangulation of M with an orbi-isomorphism from M to $\mathrm{cl}(M - (A \times I))$. Let T_1 be the triangulation of $\mathrm{cl}(M - (A \times I))$. Triangulate $A \times I$ by triangulating each $e_A \times I$ as indicated in Figure 5.1, where $e_A \times 1$ is a 2-simplex of T_1 in $|A \times 1|$. Let T_2 be the triangulation of $A \times I$. Since T_2 is compatible with T_1 on $A \times 1$, these give a triangulation of M. Let K_M be the triangulation of M.

We can easily construct an orbi-map $h : X \to X$ such that h is orbi-homotopic to the identity orbi-map id_X and that $\bar{h}(x, s) = (x, 0)$, $(x, s) \in e_A \times I$. Since $f = f \circ \mathrm{id}_X$ is orbi-homotopic to $f \circ h$ and this orbi-homotopy fixes $f|A$, we may assume that for any 2-simplex e_A of K_M in $|A|$, $\bar{f}|e_A$ is a homeomorphism from e_A to a simplex of K_N in $|B|$, and $\bar{f}(x, s) = (x, 0)$, $(x, s) \in e_A \times I$.

Let K_A be the restriction of K_M to A and K_B the restriction of K_N to B. It is clear that for each $v \in K_A^{(0)}$, $\bar{f}(v) \in T_B^{(0)}$ and $\bar{f}(O(v, K_M)) \subset O(\bar{f}(v), K_N)$, where $O(v, K)$ is the open star neighborhood of v with respect to K.

Put $K_M^* = \mathrm{Sd}_{K_A} K_M$ (the barycentric subdivision of K_M which is invariant on K_A). Put $L = \{e \in K_M^* \mid e \cap |A| = \varnothing\}$. There exists a subdivision L^* of L such that for each $v \in (L^*)^{(0)}$, there is a $w \in K_N^{(0)}$ such that $\bar{f}(O(v, L^*)) \subset O(w, K_N)$. It is easy to construct a subdivision K'_M of K_M^* such that K'_M coincides with L^* on $|L|$ and is invariant on K_A. Take $v \in (K'_M)^{(0)}$. In

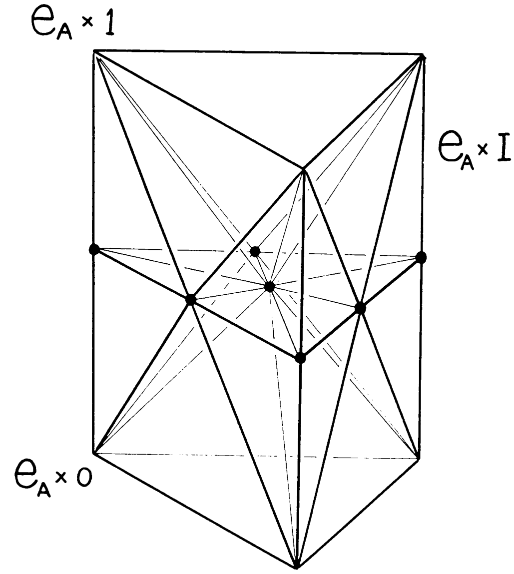


FIGURE 5.1

case $v \in \text{Int}(|L|)$. Since $O(v, K'_M) = O(v, L^*)$, $\bar{f}(O(v, K'_M)) \subset O(w, K_N)$ for some $w \in K_N^{(0)}$. In case $v \in |M| - \text{Int}(|L|)$. Since there is a vertex $u \in K_A^{(0)}$, $O(v, K'_M) \subset O(u, K_M)$, $\bar{f}(O(v, K'_M)) \subset \bar{f}(O(u, K_M)) \subset O(\bar{f}(u), K_N)$. Hence we have constructed a simplicial division K'_M of M which satisfies the following conditions; for any $v \in (K'_M)^{(0)}$, there is a vertex $w \in K_N^{(0)}$ such that $\bar{f}(O(v, K'_M)) \subset O(w, K_N)$. Namely, by the construction, for any vertex v of K'_M in $|A|$, $\bar{f}(O(v, K'_M)) \subset O(\bar{f}(v), K_N)$. Hence, in (I) of the proof of 5.1, we can choose $f(v)$ as $w(v)$. Under such a choice, the orbi-homotopy constructed in the proof of 5.1 fixes $f \mid A$. Q.E.D.

Let M be a 3-orbifold and S a spherical suborbifold which is a cyclical type (i.e. $\pi_1(S) \cong \mathbf{Z}_n$) and bounds the cone in M. Let D and D' be discal orbifolds such that $D \cap D' = \partial D = \partial D'$ and $D \cup D' = S$. It is easy to verify that D and D' are ambient orbi-isotopic in M. Hence, we have

5.3. **Proposition.** *Let M be a 3-orbifold and F, G be 2-suborbifolds of M. Suppose there exist discal suborbifolds $D \subset F$ and $D' \subset G$ such that $F - \text{Int}(D) = G - \text{Int}(D')$ and $D \cup D'$ bounds the cone in M. Then F and G are ambient orbi-isotopic in M.*

Let M be a compact 3-orbifold and X an orbifold. We say that two orbi-maps $f, g : M \to X$ are *C-equivalent* if there are orbi-maps $f = f_0, f_1, \ldots, f_n = g$ from M to X with either f_i is orbi-homotopic to f_{i-1} or f_i agrees with f_{i-1} on $M - B$ for some ballic 3-orbifold $B \subset M$ with $B \cap \partial M$ a discal orbifold or $|B| \cap |\partial M| = \varnothing$. Note that C-equivalent orbi-maps induce the same homomorphisms $\pi_1(M) \to \pi_1(X)$ up to choices of base points and inner automorphisms.

For a triangulable orbifold X, the Euler number $\chi(X)$ is defined by the formula

$$\chi(X) = \Sigma_{e_i}(-1)^{\dim(e_i)}(1/\#(G_{e_i})),$$

where e_i ranges over simplices of the given triangulation and G_{e_i} is the local group of any point of $x \in \mathrm{Int}(e_i)$. This does not depend on the choices of triangulations.

5.4. **Lemma.** *Let X and Y be orbifolds. Let $p : \widetilde{X} \to X$ and $q : \widetilde{Y} \to Y$ be the universal coverings. Suppose $\dim \widetilde{X} = n$ and $\pi_{n-1}(|\widetilde{Y}|) = 0$. Let $\tilde{g} : |\widetilde{X}| \to |\widetilde{Y}|$ be a map which satisfies the condition that there is a homomorphism $\varphi : \mathrm{Aut}(\widetilde{X}, p) \to \mathrm{Aut}(\widetilde{Y}, q)$ such that for $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, $\tilde{g} \circ \sigma = \varphi(\sigma) \circ \tilde{g}$. Then there exists a map $\tilde{f} : |\widetilde{X}| \to |\widetilde{Y}|$ which satisfies the following conditions:*

(1) *There exists a point $\tilde{x} \in |\widetilde{X}| - p^{-1}(\Sigma X)$ such that $\tilde{f}(\tilde{x}) \in |\widetilde{Y}| - q^{-1}(\Sigma Y)$.*

(2) *There exists an n-ball $B^n \subset |\widetilde{X}| - p^{-1}(\Sigma X)$ such that $B^n \cap \sigma(B^n) = \varnothing$, $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, $\sigma \neq \mathrm{id}$ and*

$$\tilde{f} \,\Big|\, \left(|\widetilde{X}| - \bigcup_{\sigma \in \mathrm{Aut}(\widetilde{X}, p)} \sigma(B^n) \right) = \tilde{g} \,\Big|\, \left(|\widetilde{X}| - \bigcup_{\sigma \in \mathrm{Aut}(\widetilde{X}, p)} \sigma(B^n) \right).$$

(3) *For each $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, $\tilde{f} \circ \sigma = \varphi(\sigma) \circ \tilde{f}$.*

Proof. If there is a point $\tilde{x} \in |\widetilde{X}| - p^{-1}(\Sigma X)$ such that $\tilde{g}(\tilde{x}) \in |\widetilde{Y}| - q^{-1}(\Sigma Y)$, we have nothing to do. Suppose there are no such points. We can take a small n-ball $B^n \subset |\widetilde{X}| - p^{-1}(\Sigma X)$ such that $B^n \cap \sigma(B^n) = \varnothing$, $\sigma \in \mathrm{Aut}(\widetilde{X}, p)$, $\sigma \neq \mathrm{id}$. Put $C = \{\sigma(B^n); \sigma \in \mathrm{Aut}(\widetilde{X}, p)\}$. Define $\tilde{f} \mid (|\widetilde{X}| - C) = \tilde{g} \mid (|\widetilde{X}| - C)$. Let A be an arc properly embedded in B^n. Let A' be an arc in ∂B^n with $\partial A = \partial A'$. Let D^2 be a disc in B^n bounded by $A \cup A'$. Take a $\tilde{x} \in \mathrm{Int}\, A$. Define $\tilde{f}(\tilde{x}) = \tilde{y}$, where $\tilde{y}$ is a point in $|\widetilde{Y}| - q^{-1}(\Sigma Y)$. Extend $\tilde{f}$ to A by using any paths in $|\widetilde{Y}|$ from $\tilde{f}(\partial A)$ to $\tilde{f}(\tilde{x})$. Since $|\widetilde{Y}|$ is simply connected, we can extend $\tilde{f}$ to D^2. Since $\pi_{n-1}(|\widetilde{Y}|) = 0$, we can extend $\tilde{f}$ to B^n. Define $\tilde{f} \mid \sigma(B^n) = \varphi(\sigma) \circ (\tilde{f} \mid B^n) \circ \sigma^{-1}$. Thus, we get the desired map $\tilde{f}$. Q.E.D.

The proof of the following theorem is modelled on the proof of 6.5 of [H].

5.5. **Theorem** (Transversal modification of dimension 3). *Suppose M and N are compact and orientable 3-orbifolds such that N is containing a properly embedded, 2-sided, 2-suborbifold F such that $\mathrm{Ker}(\pi_1(F) \to \pi_1(N)) = 1$, F is not a spherical orbifold, and the underlying space of the universal covering orbifold of each component of $\mathrm{Int}(N - F)$ is homeomorphic to $\mathbf{R}^3$. For any orbi-map $f : M \to N$, there exists an orbi-map $g : M \to N$ such that*

(1) *g is C-equivalent to f,*

(2) *each component of $g^{-1}(F)$ is a properly embedded, 2-sided, incompressible 2-suborbifold in M, and*

(3) *for properly chosen product neighborhoods $F \times [-1, 1]$ of $F = F \times 0$ in N and $g^{-1}(F) \times [-1, 1]$ of $g^{-1}(F) = g^{-1}(F) \times 0$ in M, $\bar{g}$ maps each*

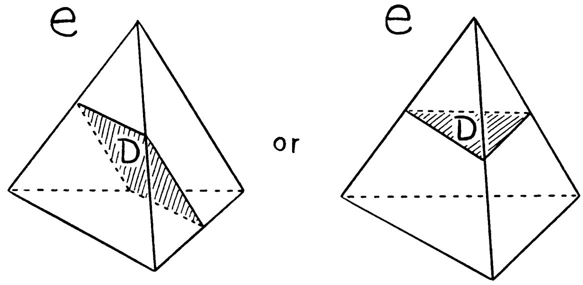


FIGURE 5.2

fiber $x \times |[-1, 1]|$ *homeomorphically to the fiber* $\bar{g}(x) \times |[-1, 1]|$ *for each* $x \in |g^{-1}(F)|$.

Proof. We suppose that N is triangulated in such a way that a product neighborhood $F \times [-1, 1]$ of F is triangulated as a product. That is, each simplex in $F \times [-1, 1]$ is the join of a simplex in $F \times 1$ with a simplex in $F \times (-1)$. Then, we may assume that $|F| = |F \times 0|$ possesses the following properties with respect to the triangulation K_N of N; (a) $|F| \cap K_N^{(0)} = \varnothing$, (b) $|F|$ intersects transversely with $|K_N^{(1)}|$, (c) for a 3-simplex $e \in K_N^{(3)}$, $e \cap |F|$ is either empty or topologically only one 2-cell D which satisfies the following properties; if for an edge $e^{(1)}$ of e, $e^{(1)} \cap D \neq \varnothing$, then $e^{(1)} \cap D$ is a point and if for a face $e^{(2)}$ of e, $e^{(2)} \cap D \neq \varnothing$, then $e^{(2)} \cap D$ is an arc properly embedded in $e^{(2)}$ (See Figure 5.2). By 5.1, we can suppose that M is triangulated in such a way that f is a simplicial orbi-map with respect to K_N. It is easy to check that each component of $f^{-1}(F)$ is a properly embedded, 2-sided 2-suborbifold and that condition (3) is satisfied relative to $F \times [-1/2, 1/2]$ and $f^{-1}(F \times [-1/2, 1/2])$ with no differences in the case of manifolds. If each component of $f^{-1}(F)$ is incompressible, the proof is completed. If not, we consider the following three possibilities.

Case 1. $f^{-1}(F)$ contains a compressible spherical orbifold S. Then S bounds a ballic orbifold B in M. Let U be a small regular neighborhood B. Put $\partial U = S \times t_0 \subset S \times [-1, 1]$. (See Figure 5.3.) From the hypothesis, by Lemma 4.3, there exists an orbi-map $f' : B \to F$ such that $f' | \partial B = f | S$. If necessary, by using 5.4, we can define an orbi-map $f_1 : M \to N$ as follows;

(a) $(f_1 | M - \mathrm{Int}(U)) = (f | M - \mathrm{Int}(U))$, (if necessary, by using 5.4, we may assume that there is a point $x \in |M| - (|\mathrm{Int}\, U| \cup \Sigma M)$ such that $\bar{f}(x) \in |N| - \Sigma N$.)

(b) $(f_1 | S \times [0, t_0]) : S \times [0, t_0] \to F \times t_0$, by $f \times c$,

(c) $(f_1 | B) : B \to F \times t_0$, by $f' \times c$, where $c : [0, t_0] \to t_0$ is a constant map. (Note that we regard c as the orbi-map both of whose underlying map and structure map are the constant map $[0, t_0] \to t_0$.)

Then f_1 is C-equivalent to f and $f_1^{-1}(F) \subset f^{-1}(F) - S$.

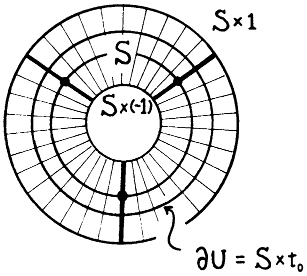


FIGURE 5.3

Case 2. $f^{-1}(F)$ contains a compressible discal orbifold D. As in case 1, we obtain an orbi-map $f_1 : M \to N$ C-equivalent to f with $f_1^{-1}(F) \subset f^{-1}(F) - D$.

Case 3. There is a discal orbifold D in $\mathrm{Int}(M)$ with $D \cap f^{-1}(F) = \partial D$ and ∂D does not bound any discal orbifolds in $f^{-1}(F)$. Let U be a regular neighborhood of D in M such that $A = U \cap f^{-1}(F)$ is an annulus properly embedded in U. Let D_1 and D_2 be the disjoint discal orbifolds in ∂U with $\partial A = \partial D_1 \cup \partial D_2$, and choose disjoint discal orbifolds E_1 and E_2 properly embedded in U with $\partial E_i = \partial D_i$ and $|E_i| \cap |D| = \varnothing$. (See Figure 5.4.) We may assume that $\bar{f}(|\partial E_i|) \subset |N| - \Sigma N$ (if necessary, by modifying with an orbi-homotopy fixing out of a small tubular neighborhood of $|\partial E_i|$). We define an orbi-map $f_1 : M \to N$ as follows: Put $f_1 \mid (M - \mathrm{Int}(U)) = f \mid (M - \mathrm{Int}(U))$. Since $[|\partial E_i|]^n = 1$ in $\pi_1(M)$ for some $n \in \mathbf{N}$, $[\bar{f}_1(|\partial E_i|)]^n = 1$ in $\pi_1(N)$. By 3.3, we can extend $f_1 \mid \partial E_i$ to orbi-maps from E_i to F. We can find ballic orbifolds B_i, B_i', C_1, C_2, and C_3 indicated as Figure 5.5 with $B_i = E_i \times [0, 1]$ and $B_i' = E_i \times [-1, 0]$. Define an orbi-map $(f_1 \mid B_i^{(\prime)}) : E_i \times [0, \varepsilon] \to F \times [0, \varepsilon]$ by $(f_1 \mid B_i) = (f_1 \mid E_i) \times \mathrm{id}$, $x \in E_i$, where $\varepsilon = \pm 1$. Put $S_i = \partial C_i$. S_i is a spherical orbifold. Since $f_1(S_i) \subset \mathrm{Int}(N - F)$, by 4.1, f_1 is extendable to an orbi-map from C_i to $N - F$. Let A' be an annulus in ∂U such that $\partial A' = \partial E_1 \times (-1) \cup \partial E_2 \times (-1)$. Put $S' = A' \cup (E_1 \times (-1)) \cup (E_2 \times (-1))$ $(= \partial C_3)$. For the same reason, we can extend $f_1 \mid S'$ to an orbi-map from C_3 to $N - F$. Then f_1 is C-equivalent to f and $f_1^{-1}(F) = (f^{-1}(F) - A) \cup E_1 \cup E_2$. Note that f_1 is forced to an orbi-map opposite sides of E_i to opposite side of F.

In each of the above cases, f_1 satisfies (3) of the theorem.

We define the complexity of f, denoted by $c(f)$, as follows; Let L be the maximum of the orders of the local groups of M. Suppose the minimal number of the Euler numbers of all the components of $f^{-1}(F)$ is more than $-r + (m-1)/L$ and is not more than $-r + m/L$, where $r \in \mathbf{Z}$ and $m \in \{0, 1, 2, \ldots, L-1\}$. Let $n_{-r+i+j/L}$ be the number of the components of $f^{-1}(F)$ whose Euler numbers are more than $-r + i + (j-1)/L$ and not more

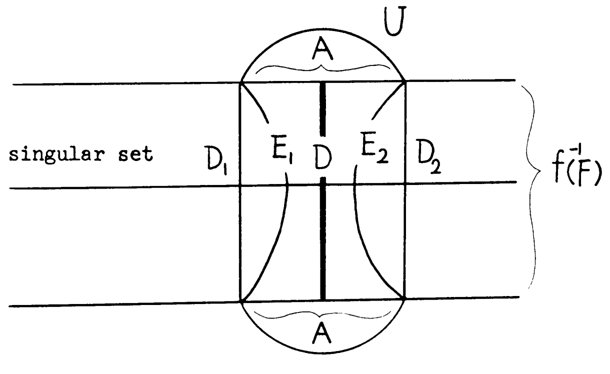


FIGURE 5.4

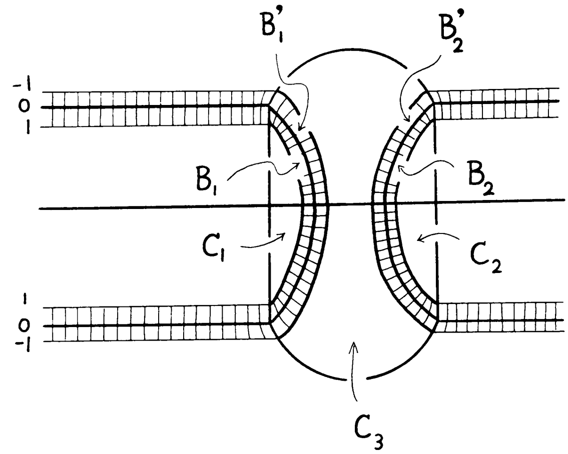


FIGURE 5.5

than $-r+i+j/L$. We define

$$c(f)=(n_{-r+m/L}, n_{-r+(m+1)/L}, \dots, n_{-r+1}, n_{-r+1+m/L}, \dots, n_2).$$

Let $\mathscr{M}$ be the set of all orbi-maps $f: M \to N$ such that the minimum of Euler number of all components of $f^{-1}(F)$ is not less than $-r+m/L$. We order $c(\mathscr{M})$ lexicographically: i.e.,

$$(n_{-r+m/L}, \dots, n_2) < (n'_{-r+m/L}, \dots, n'_2)$$

if there is some $t \in \{-r+m/L, -r+(m+1)/L, \dots, 2\}$ such that

$$n_k = n'_k \quad \text{for } k<t \quad \text{and} \quad n_t < n'_t.$$

It is easy to verify that in every case $c(f_1) < c(f)$. Thus, by an inductive argument, the proof will be completed. Q.E.D.

5.6. **Corollary.** *Let M be a compact and orientable 3-orbifold. If there is an epimorphism $\varphi: \pi_1(M) \to \mathbf{Z}$, then there exists a properly embedded, 2-sided, nonseparating incompressible 2-suborbifold in M.*

Proof. Let S be the solid torus. By 4.2, there exists an orbi-map $f : M \to S$ such that $f_* = \varphi$. Let D be a meridian disc of S. By 5.5, each component of $f^{-1}(D)$ is a properly embedded, 2-sided, incompressible 2-suborbifold in M. Since f_* is epimorphism, there is a loop α in $|M| - \Sigma M$ such that $f_*[\alpha] = z$; the generator of $\pi_1(S)$. By (3) of 5.5, there exists a component F of $f^{-1}(S)$ such that α meets $|F|$ transversely in nonzero intersection number. Hence, there exists a loop in $|M| - \Sigma M$ which intersects F transversely in some direction at least one. This implies F is nonseparating. Q.E.D.

Let F be a 2-orbifold. A simple closed curve C in $|\operatorname{Int}(F)| - \Sigma F$ is called *essential* if C does not bound the underlying space of any discal orbifold in F. A simple arc A properly embedded in $|F| - \Sigma F$ is called *essential* if there is a subarc B of $|\partial F|$ such that $A \cup B$ does not bound the underlying space of any 2-discs in F. Let B be a essential simple closed curve or a essential simple arc. Let F' be a component of $\operatorname{cl}(F - B \times I)$. Let $p : \widetilde{F} \to F$ be the finite uniformization, $\widetilde{F}'$ a component of $p^{-1}(F')$, and $\widetilde{B}$ a lift of B. It is easy to see that $\widetilde{B}$ is essential (in the ordinary sense) in $|\widetilde{F}|$. Hence $\pi_1(\widetilde{F}') \to \pi_1(\widetilde{F})$ is monic. By the same way as in the proof of 3.9(b), $\pi_1(F') \to \pi_1(F)$ is monic.

5.7. **Lemma.** *Suppose F and G are compact and orientable 2-orbifolds. Let f, g be two orbi-maps from F to G and D a discal suborbifold in F such that $(f \mid F - \operatorname{Int}(D)) = (g \mid F - \operatorname{Int}(D))$. Then f and g are orbi-homotopic.*

Proof. Define an orbi-map $F : F \times I \to G$ as follows: $F \mid ((F - \operatorname{Int}(D)) \times I) = (f \mid (F - \operatorname{Int}(D))) \times \mathrm{id}$, $F \mid (D \times 0) = f \mid D$, and $F \mid (D \times 1) = g \mid D$. By 4.3, we can extend F to the remaining parts. It is clear that $F \mid (F \times 0) = f$ and $F \mid (F \times 1) = g$. Q.E.D.

Let L be a 1-suborbifold in a 2-orbifold. L is called *essential* if $|L|$ is essential.

5.8. **Theorem** (Transversal modification of dimension 2). *Suppose F and G are compact and orientable 2-orbifolds such that G is containing a properly embedded, 2-sided 1-suborbifold L with* $\operatorname{Ker}(\pi_1(L) \to \pi_1(G)) = 1$. *If $f : F \to G$ is any orbi-map, then there exists an orbi-map $g : F \to G$ such that*

(1) *g is orbi-homotopic to f,*

(2) *each component of $g^{-1}(L)$ is a properly embedded, 2-sided, essential 1-suborbifold in F, and*

(3) *for properly choosen product neighborhoods $L \times [-1, 1]$ of $L = L \times 0$ in G and $g^{-1}(L) \times [-1, 1]$ of $g^{-1}(L) = g^{-1}(L) \times 0$ in F, $\bar{g}$ maps each fiber $x \times |[-1, 1]|$ homeomorphically to the fiber $\bar{g}(x) \times |[-1, 1]|$ for each $x \in |g^{-1}(L)|$.*

Proof. By the same principle of 5.5, we may assume that $f^{-1}(L)$ is a properly embedded, 2-sided, 1-suborbifold in F and that (3) holds. If each component of $f^{-1}(L)$ is essential, the proof is completed. If not, we consider the following two possibilities.

Case 1. A component C of $f^{-1}(L)$ bounds a discal orbifold D in F. Let U be the regular neighborhood of D in F. Put $\partial U = C \times t_0 \subset C \times [-1, 1]$. Since $[|C|]^n = 1$ in $\pi_1(F)$ for some $n \in \mathbf{N}$, $[\bar{f}(|C|)]^n = 1$ in $\pi_1(G)$. Since $\mathrm{Ker}(\pi_1(L) \to \pi_1(G)) = 1$, $[\bar{f}(|C|)]^n = 1$ in $\pi_1(L)$. Hence $[\bar{f}(|C|)] = 1$ in $\pi_1(L)$. Hence $(f \mid C) : C \to L$ is extendable to an orbi-map $(f' \mid D) : D \to L$. Define an orbi-map $f_1 : F \to G$ in a manner similar to the proof of 5.4. By 5.7, f_1 is orbi-homotopic to f and $f_1^{-1}(L) \subset f^{-1}(L) - C$.

Case 2. The underlying space of a component B of $f^{-1}(L)$ is an inessential arc in $|F| - \Sigma F$. As in the proof of 5.7 and Case 1, we obtain an orbi-map $f_1 : F \to G$ orbi-homotopic to f with $f_1^{-1}(L) \subset f^{-1}(L) - B$.

We define the complexity $c(f)$ of f by the number of the components of $f^{-1}(L)$. Clearly $c(f_1) < c(f)$. Thus the proof will be completed by an inductive argument. Q.E.D.

6. I-BUNDLES

6.1. **Theorem.** *Let M be a compact and orientable 3-orbifold with boundaries and F ($\neq$ spherical orbifold) a component of ∂M such that $i_* : \pi_1(F) \to \pi_1(M)$ is an isomorphism. Then M is orbi-isomorphic to $F \times I$ by an orbi-isomorphism which takes F to $F \times 0$.*

Proof. [B-N and F] show that good orientable 2-orbifolds are finitely uniformizable. Thus there exists a torsion free normal subgroup G of $i_*\pi_1(F)$ ($\cong \pi_1(M)$). Let $p : \widetilde{M} \to M$ be the finite uniformization with $p_*\pi_1(\widetilde{M}) = G$. Let $\widetilde{F}$ be a component of $p^{-1}(F)$. Note that $\pi_1(\widetilde{F})$ injects into G. Let $j : \widetilde{F} \to \widetilde{M}$ be the inclusion orbi-map. Suppose $|G; p_*j_*\pi_1(\widetilde{F})| > 1$; then

$$\begin{aligned}
&|\pi_1(M); p_*j_*\pi_1(\widetilde{F})| \\
&\quad = |\pi_1(M); p_*\pi_1(\widetilde{M})| \times |p_*\pi_1(\widetilde{M}); p_*j_*\pi_1(\widetilde{F})| \\
&\quad > |\pi_1(M); p_*\pi_1(\widetilde{M})| = \text{the number of sheets of } p.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\pi_1(M); p_*j_*\pi_1(\widetilde{F})| &= |i_*\pi_1(F); p_*j_*\pi_1(\widetilde{F})| \\
&= |\pi_1(F); i_*^{-1}p_*j_*\pi_1(\widetilde{F})| \\
&= \text{the number of sheets of } (p \mid \widetilde{F}) : \widetilde{F} \to F.
\end{aligned}$$

Since the number of sheets of $(p \mid \widetilde{F})$ cannot be greater than the number of sheets of p, this is a contradiction. Hence $G = p_*j_*\pi_1(\widetilde{F})$, and so by 2.4, $\pi_1(\widetilde{M}) = j_*\pi_1(\widetilde{F})$. By Theorem 10.2 of [H], $\widetilde{M} = \widetilde{F} \times I$. Since

$$\begin{aligned}
|\pi_1(M); p_*\pi_1(\widetilde{M})| &= |i_*\pi_1(F); p_*j_*\pi_1(\widetilde{F})| \\
&= |\pi_1(F); (p \mid \widetilde{F})_*\pi_1(\widetilde{F})|,
\end{aligned}$$

$p^{-1}(F) = \widetilde{F}$. Consequently, ∂M consists of two components. By Theorem 8.1 of [M-S], we have the conclusion. Q.E.D.

6.2. **Theorem.** *Let M be a compact and orientable 3-orbifold with boundaries and F ($\neq$ spherical orbifold) a component of ∂M which is incompressible in M. If the index, $|\pi_1(M); i_*\pi_1(F)|$, of $i_*\pi_1(F)$ in $\pi_1(M)$ is finite, then either*

(1) $|\pi_1(M); i_*\pi_1(F)| = 1$ *and* $M = F \times I$ *with* $F = F \times 0$, *or*

(2) $|\pi_1(M); i_*\pi_1(F)| = 2$ *and M is a twisted I-bundle over a compact nonorientable closed 2-orbifold.*

Proof. Let $p : \widetilde{M} \to M$ be the covering associated with $i_*\pi_1(F)$. There is a component $\widetilde{F}_0$ of $\partial\widetilde{M}$ such that $(p\,|\,\widetilde{F}_0) : \widetilde{F}_0 \to F$ is an isomorphism. Since $\pi_1(\widetilde{F}_0) \to \pi_1(\widetilde{M})$ is an isomorphism, $\widetilde{M} = \widetilde{F}_0 \times I$ by 6.1. Let $\widetilde{F}_1$ be the other boundary component of $\widetilde{F}_0 \times I$.

Case 1. $p(\widetilde{F}_1)$ is not contained in F.

We have $|\pi_1(M); i_*\pi_1(F)| = 1$. By 6.1, $M = F \times I$.

Case 2. $p(\widetilde{F}_1) \subset F$.

Let k be the number of sheets of the covering $(p\,|\,\widetilde{F}_1) : \widetilde{F}_1 \to F$. Then

$$k + 1 = |\pi_1(M); i_*\pi_1(F)|.$$

On the other hand, $k = |\pi_1(F); p_*\pi_1(\widetilde{F}_1)| = 1$. Hence $|\pi_1(M); i_*\pi_1(F)| = 2$. By Theorem 8.1 of [M-S], M is a twisted I-bundle over a nonorientable closed 2-orbifold. Q.E.D.

6.3. **Theorem.** *Let M be a compact and orientable 3-orbifold. Suppose that $\pi_1(M)$ contains a subgroup G of finite index which is isomorphic to the orbifold fundamental group of some closed and orientable 2-orbifold ($\neq$ spherical orbifold). Then M is an I-bundle over some closed 2-orbifold.*

Proof. Let $p_1 : \widetilde{M} \to M$ be the covering associated with G. There is a torsion free normal subgroup G' of G with finite index in G. Let $p_2 : M' \to \widetilde{M}$ be the uniformization associated with G'. Since $\pi_1(M') \cong G'$, by 10.6 of [H], M' is a product I-bundle over a closed, orientable 2-manifold whose fundamental group is isomorphic to G'. Since $p_1 \circ p_2 : M' \to M$ is a finite covering, by 8.1 of [M-S], we have the conclusion. Q.E.D.

7. The classification theorem

7.1. **Lemma.** *Let $f : D^2(m) \to D^2(n)$ be an orbi-map such that $f\,|\,\partial D^2(m)$ is an orbi-covering from $\partial D^2(m)$ to $\partial D^2(n)$. Suppose m divides n. Then there exists an orbi-homotopy $F : D^2(m) \times [0, 1] \to D^2(n)$ such that $F\,|\,(D^2(m) \times 0) = f$, $F\,|\,(D^2(m) \times 1)$ is an orbi-covering from $D^2(m) \times 1$ to $D^2(n)$ and $F\,|\,(\partial D^2(m) \times t) = f\,|\,\partial D^2(m)$ for any $t \in |[0, 1]|$.*

Proof. Let $\bar{f}$ and $\tilde{f}$ be the underlying map and the structure map of f, respectively. Let v be a singular point of $D^2(m)$ and u a singular point of $D^2(n)$. Let $p : D^2 \to D^2(m)$ and $q : D^2 \to D^2(n)$ be the universal coverings. Put $\tilde{v} = p^{-1}(v)$ and $\tilde{u} = q^{-1}(u)$. Define a map $\bar{g} : |D^2(m)| \to |D^2(n)|$

by $\bar{g}((1-t)v+tx)=(1-t)u+\bar{f}(x)$, where $x \in |\partial D^2(m)|$. Define a map $\tilde{g}:|D^2| \to |D^2|$ by $\tilde{g}((1-t)\tilde{v}+t\tilde{x})=(1-t)\tilde{u}+t\tilde{f}(\tilde{x})$, where $\tilde{x} \in |\partial D^2|$. We can define an orbi-map $g: D^2(m) \to D^2(n)$ by $g=(\bar{g}, \tilde{g})$. Define an orbi-map $F':\partial(D^2(m)\times[0,1]) \to D^2(n)$ by $F'|(D^2(m)\times 0)=f$, $F'|(D^2(m)\times 1)=g$, and $F'|(\partial D^2(m)\times[0,1])=f|\partial D^2(m)$. By 4.3, F' is extendable to an orbi-map from $D^2(m)\times[0,1]$ to $D^2(n)$. Q.E.D.

A 2-orbifold F is called a *turnover* if F is a 2-sphere and ΣF consists of three points.

The proofs of 7.2 and 7.4 are modeled on the proofs of 13.1 and 13.6 of [H].

7.2. **Theorem.** *Let F and G be compact, orientable 2-orbifolds such that $\#\pi_1(F)=\infty$ and G is not a turnover. Suppose $f:(F,\partial F)\to(G,\partial G)$ is an orbi-map such that $f_*:\pi_1(F)\to\pi_1(G)$ is monic. Then there exists an orbi-homotopy $f_t:(F,\partial F)\to(G,\partial G)$, $t\in|I|$, with $f_0=f$ and either*

(1) $f_1:F\to G$ is an orbi-covering, or

(2) F is an annulus and $f_1(F)\subset\partial G$.

If for some component J of ∂F, $(f|J):J\to f(J)$ is an orbi-covering, we can require $(f_t|J)=(f|J)$ for all t.

Proof. In case $|\partial G|\neq\varnothing$. Suppose $|\partial F|=\varnothing$. Since F is finitely uniformizable, there is a fundamental group of a closed orientable surface S ($\neq S^2$), as a subgroup of $\pi_1(F)$. Hence $\pi_1(S)\cong f_*\pi_1(S)<\pi_1(G)$. On the other hand, since $|\partial G|=\varnothing$, $\pi_1(G)=\mathbf{Z}*\cdots*\mathbf{Z}*(\mathbf{Z}/m_1)*\cdots*(\mathbf{Z}/m_r)$. By the Kurosh Subgroup Theorem (8.3 of [H]), this is a contradiction. Hence $|\partial F|\neq\varnothing$.

Let J be any component of ∂F and K a component of ∂G such that $f(J)\subset K$. Since the maps $\pi_1(J)\to\pi_1(F)$ and $f:\pi_1(F)\to\pi_1(G)$ are monic, $(f|J)_*:\pi_1(J)\to\pi_1(K)$ is monic. Thus, after modifying by an orbi-homotopy, we may assume $(f|J):J\to K$ is an orbi-covering. We do this for each component of ∂F.

Let $q:G'\to G$ be a covering with $q_*\pi_1(G')=f_*\pi_1(F)$. Let $f':F\to G'$ be a lift of f with respect to q. Since $q\circ f'=f$, by 2.1, $q_*\circ f'_*=f_*$. Hence f'_* is monic. Since $q_*\pi_1(G')=f_*\pi_1(F)=(q_*\circ f'_*)(\pi_1(F))$ and q_* is monic, f'_* is epic. Thus, f'_* is an isomorphism. Note that for each component J of ∂F, there is a component K' of $\partial G'$ such that $(f'|J):J\to K'$ is an orbi-covering.

Suppose that $f'|\partial F$ is not an orbi-embedding. With this assumption, in the same way as in the proof of manifolds (see p. 138 of [H]), we can show that there is a path $\alpha:(I,\partial I)\to(|F|-\Sigma F,|\partial F|)$ satisfying

(*) $\alpha(0)\neq\alpha(1)$, $\bar{f}'(\alpha(0))=\bar{f}'(\alpha(1))\in|\partial G'|$, and $[\tilde{f}'\circ\tilde{\alpha}]=1$ in $\pi_1(G')$,

where $\tilde{\alpha}$ is a lift of α in the underlying space of the universal covering orbifold of F, $\tilde{f}'$ is the structure map of f', and $\bar{f}'$ is the underlying map of f'. Let J_i be the component of ∂F with $x_i=\alpha(i)\in|J_i|$ $(i=0,1)$ and let K be the component of $\partial G'$ with $y=\bar{f}'(x_0)=\bar{f}'(x_1)\in|K|$. Let $\eta_0:\pi_1(J_0,x_0)\to\pi_1(F,x_0)$ be the homomorphism induced by the inclusion orbi-map. Let $p:\widetilde{F}\to F$ be the covering associated with $\eta_0\pi_1(J_0,x_0)$. Let $\hat{\alpha}$

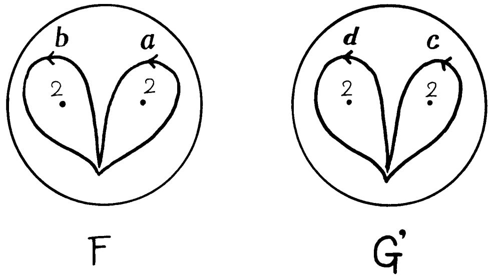


FIGURE 7.1

be the lifting of α with $\hat{\alpha}(0) = \hat{x}_0$. Put $\hat{x}_1 = \hat{\alpha}(1)$ and let $\widetilde{J}_i$ $(i = 0, 1)$ be the component of $p^{-1}(J_i)$ with $\hat{x}_i = |\widetilde{J}_i|$. Since $\eta_0 \pi_1(J_0, x_0) \cong \mathbf{Z}$ is torsion free, $\widetilde{F}$ is a manifold. In the same way as in the proof of manifolds (see pp. 138–139 of [H]), we can conclude that $\widetilde{F}$ is an annulus and $\widetilde{J}_1 \neq \widetilde{J}_0$. Let k be the number of sheets of the covering $p : \widetilde{F} \to F$. Since $\chi(F) = (1/k)\chi(\widetilde{F}) = 0$, F is an annulus or $D^2(2, 2)$.

Suppose F is an annulus. We can construct an orbi-homotopy retracting f' into $\partial G'$, since $f'(\partial F) \subset \partial G'$, $[\tilde{f}' \circ \tilde{\alpha}] = 1$ in $\pi_1(G')$, and G' is not a spherical 2-orbifold. Hence so is f. This implies conclusion (2) holds.

Suppose F is $D^2(2, 2)$. Note the underlying map $\bar{f}'$ is a map between the underlying spaces of F and G'. Since $(\bar{f} \,|\, |\partial F|) : |\partial F| \to |K|$ is a covering map and $\bar{f}'$ is extendable to a map from a 2-disc $|F|$ to $|G'|$, $[\bar{f}' \,|\, |\partial F|] = [|K'|]^m = 1$ in $\pi_1(|G'|)$ for some $m \in \mathbf{Z} - \{0\}$. Hence $|G'|$ is a 2-disc. Since f'_* is an isomorphism, G' must be $D^2(2, 2)$. Take the generators of $\pi_1(F)$ and $\pi_1(G')$ indicated in Figure 7.1. Since any subgroup of $D^2(2, 2)$ generated by a multiple of the boundary loop is normal, we may assume that $f'_*(ab) = (cd)^m$ for some $m \in \mathbf{Z} - \{0\}$. Let $\langle ab \rangle$ be the normal subgroup of $\pi_1(F)$ generated by ab and $\langle (cd)^m \rangle$ the normal subgroup of $\pi_1(G')$ generated by $(cd)^m$. Since f'_* is an isomorphism, $\pi_1(F)/\langle ab \rangle \cong \pi_1(G')/\langle (cd)^m \rangle$. On the other hand, $\pi_1(F)/\langle ab \rangle \cong \mathbf{Z}_2$ and $\pi_1(G')/\langle (cd)^m \rangle \cong$ the dihedral group of order $2|m|$. Then, it must be $m = \pm 1$. Hence, $f'_*([|\partial F|]) = [|K|]^{\pm 1}$ in $\pi_1(G')$. This contradicts the fact that $f' \,|\, \partial F$ is not an orbi-embedding.

Suppose that $(f' \,|\, \partial F)$ is an orbi-embedding. Consider the diagram of the exact sequence of homology.

$$\begin{array}{ccccccccc}
\to H_2(|F|) & \to & H_2(|F|, \partial|F|) & \xrightarrow{\partial_*} & H_1(\partial|F|) & \to & H_1(|F|) & \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\to H_2(|G'|) & \to & H_2(|G'|, \partial|G'|) & \xrightarrow[\partial_*]{} & H_1(\partial|G'|) & \to & H_1(|G'|) & \to
\end{array}$$

Since $H_2(|F|) = H_2(|G'|) = 0$, ∂_* is monic. Since $(f' \,|\, \partial F)$ is an orbi-embedding, $\bar{f}'_* \partial_*([|F|]) = \bar{f}'_*([|\partial F|]) \neq 0 \in H_1(\partial|G'|)$. Hence $H_2(|G'|, \partial|G'|) \neq 0$. This implies that $|G'|$ is compact. That is, G' is compact. Since G' is compact, $|\partial G'| \neq \emptyset$ and $\pi_1(G') \cong \pi_1(F)$ is not finite, there exist 1-suborbifolds

$A_1, \dots, A_m$ whose underlying spaces are essential simple arcs properly embedded in $|G'| - \Sigma G'$ such that each component of $\mathrm{cl}(G' - \{A_i\} \times I)$ is a discal orbifold.

By 5.1 and 5.2, by modifying f' under an orbi-homotopy which fixes boundaries, we may assume that each component of $\{f'^{-1}(A_i)\}$ is a 1-suborbifold properly embedded in F. Let B be a component of $\{f'^{-1}(A_i)\}$. If $|B|$ is an inessential simple closed curve, then by 5.8, we can remove B by an orbi-homotopy fixing boundaries. $|B|$ must not be an essential simple closed curve since f'_* is an isomorphism. $|B|$ must not be an inessential simple arc since $\bar{f}'(B)$ is an essential arc. Thus we may assume that $|B|$ is a properly embedded, essential arc. By modifying f' under an orbi-homotopy fixing boundaries, we may assume that $(\bar{f}' \mid |B|) : |B| \to \bar{f}'(|B|)$ is a homeomorphism. Iterating such modifications, we may assume that for each component B of $\{f'^{-1}(A_i)\}$, $|B|$ is a properly embedded, essential arc and $(\bar{f}' \mid |B|) : |B| \to \bar{f}'(|B|)$ is a homeomorphism. By (3) of 5.8, for any component G'' of $\mathrm{cl}(G' - (\{A_i\} \times I))$, there is a component F' of $\mathrm{cl}(F - \{f'^{-1}(A_i)\} \times I)$ such that $f'(F') \subset G''$. Since the maps $\pi_1(F') \to \pi_1(F)$ and $f : \pi_1(F) \to \pi_1(G')$ are monic, $\pi_1(F') \to \pi_1(G'')$ is monic. Since G'' is a discal orbifold, F' is a discal orbifold. Thus, $(f' \mid F') : F' \to G''$ is an orbi-map from a discal orbifold to a discal orbifold, $(f' \mid \partial F') : \partial F' \to \partial G''$ is an orbi-covering and $(f' \mid F')_* : \pi_1(F') \to \pi_1(G'')$ is monic. Hence $F' = D^2(m)$ and $G'' = D^2(n)$, where m divides n. By 7.1, under an orbi-homotopy fixing boundaries, we can modify $f' \mid F'$ to an orbi-covering from F' to G''. By piecing together such maps, the conclusion (1) holds.

In case $|\partial G| = \varnothing$. (Automatically, $|\partial F| = \varnothing$.) Since G is not a turnover, there exists a 1-suborbifold C whose underlying space is an essential simple closed curve in $|G| - \Sigma G$. Let G' be a component of $\mathrm{cl}(G - C \times I)$. By 5.8, we may assume that the underlying space of each component of $f^{-1}(C)$ is essential simple closed curve in $|F| - \Sigma F$. By modifying f under an orbi-homotopy, we may assume that f is an orbi-covering on each component of $f^{-1}(C)$ and (3) of 5.8 stands.

Let $F_1, \dots, F_n$ be the components of $\mathrm{cl}(F - f^{-1}(C) \times I)$. By the previous case, for each $(f \mid F_i) : F_i \to G_i$, the conclusion (1) or (2) holds, where G_i is the component of $\mathrm{cl}(G - (C \times I))$ with $f(F_i) \subset G_i$. If conclusion (2) holds for one of the resulting pieces, we can reduce the number of components of $f^{-1}(C)$ by modifying f under an orbi-homotopy.

We cannot completely eliminate $f^{-1}(C)$; otherwise f_* is a monomorphism from $\pi_1(F)$ to $\pi_1(G_i)$ for some i. Since $|\partial F| = \varnothing$, there is a fundamental group of a closed orientable surface $(\neq S^2)$ as a subgroup of $\pi_1(F)$. On the other hand, $\pi_1(G - C) = \mathbf{Z} * \cdots * \mathbf{Z} * (\mathbf{Z}/m_1) * \cdots * (\mathbf{Z}/m_r)$. By the Kurosh Subgroup Theorem, this is a contradiction.

Thus, the conclusion (1) follows by fitting together the pieces. Q.E.D.

Let M be a compact 3-orbifold. A sequence

$$M = M_1 \supset M_2 \supset \cdots \supset M_n$$

of 3-orbifolds is called a *hierarchy* for M provided that M_{i+1} is obtained from M_i by cutting open along a properly embedded, 2-sided incompressible 2-suborbifold F_i and each component of M_n is a ballic 3-orbifold. The integer n is called the *length* of the hierarchy. If M has a hierarchy, then M is irreducible. (Note that, in 3.9, if each component of M' is irreducible, then M is irreducible, and that all ballic orbifolds are irreducible.)

A 2-suborbifold F in a 3-orbifold M is said to be *boundary-parallel* if one of the components of $\mathrm{cl}(M - F)$ is orbi-isomorphic to $F \times I$. A 3-orbifold M is called *sufficiently large* if there exists a 2-sided and incompressible 2-suborbifold F which is not boundary parallel.

W. D. Dunbar showed the following theorem in [D].

7.3. **Theorem** (Dunbar [D]). *Let M be a compact, irreducible and orientable 3-orbifold in which there is no turnover with nonpositive Euler number. If M is sufficiently large, then M has a hierarchy.*

Let M be a compact, orientable, and finitely uniformizable 3-orbifold. Suppose M has a hierarchy. Then, by 3.8, any compact covering orbifold $\widetilde{M}$ of M also has a hierarchy. Hence, in case M has a hierarchy of length $n \geq 2$, by 13.4 of [H], the interior of the underlying space of the universal covering orbifold of M is homeomorphic to $\mathbf{R}^3$.

Let $\mathscr{W}$ be the class of all compact and orientable 3-orbifolds which are
(i) finitely uniformizable,
(ii) irreducible,
(iii) sufficiently large,
(iv) in which there is no turnover with nonpositive Euler number.

Now we have sufficiently prepared for showing the main theorem.

7.4. **Theorem.** *Let $M, N \in \mathscr{W}$ and suppose $f : (M, \partial M) \to (N, \partial N)$ is an orbi-map such that $f_* : \pi_1(M) \to \pi_1(N)$ is monic and such that for each component B of ∂M, $(f \mid B)_* : \pi_1(B) \to \pi_1(C)$ is monic, where C is the component of ∂N with $f(B) \subset C$. Then there exists an orbi-map $g : (M, \partial M) \to (N, \partial N)$, $t \in |I|$, such that $g_* = f_* : \pi_1(M) \to \pi_1(N)$ and either*

(1) *$g : M \to N$ is an orbi-covering,*

(2) *M is an I-bundle over a closed 2-orbifold, there is an orbi-homotopy $f_t : (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$, $f_1 = g$, and $g(M) \subset \partial N$, or*

(3) *Each of M and N is* (a) *or* (b) *in Figure* 7.2, *and $g \mid \partial M$ is an orbi-covering.*

If $(f \mid B) : B \to C$ is already an orbi-covering, we may assume $(f \mid B) = (g \mid B)$, and in case (2), *$f_t \mid B = f \mid B$ for all t.*

Proof. By 7.2, we may assume, after changing f by an orbi-homotopy, that for each component B of ∂M, $(f \mid B)$ is an orbi-covering from B to a component

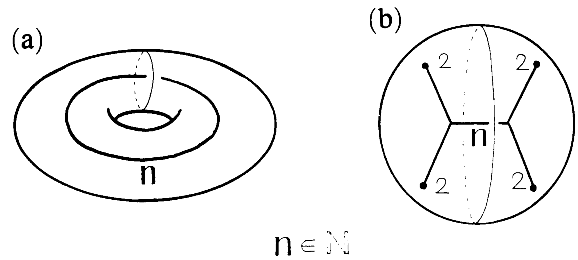


FIGURE 7.2

of ∂N. If this is already the case for some B, there is no need now nor in any future step to change $(f \mid B)$.

Let $q : N' \to N$ be a covering with $q_* \pi_1(N') = f_* \pi_1(M)$. Note that N' is also finitely uniformizable since $\pi_1(N') \cong \pi_1(M)$. Let f' be a lift of f with respect to q. By the same reason as in the proof of 7.2, f'_* is an isomorphism.

Note that for each component B of ∂M, there is a component C of $\partial N'$ such that $(f' \mid B) : B \to C$ is an orbi-covering.

In case $|\partial M| \neq \emptyset$ *and* $f' \mid \partial M$ *is not an orbi-embedding.* With this assumption, in the same way as in the proof of 3-manifolds (see p. 144 of [H]), we can show that there is a path $\alpha : (I, \partial I) \to (|M| - \Sigma M, |\partial M|)$ satisfying

$(*)$ $\alpha(0) \neq \alpha(1)$, $\bar{f}'(\alpha(0)) = \bar{f}'(\alpha(1)) \in |\partial N'| - \Sigma N'$, and $[\tilde{f}' \circ \hat{\alpha}] = 1$ in $\pi_1(N')$, where $\hat{\alpha}$ is a lift of α in the underlying space of the universal covering orbifold of M, $\tilde{f}'$ is the structure map of f', and $\bar{f}'$ is the underlying map of f'.

Let B_i be the component of ∂M with $x_i = \alpha(i) \in |B_i|$ $(i = 0, 1)$ and let C be the component of $\partial N'$ with $y = \bar{f}'(x_0) = \bar{f}'(x_1) \in |C|$.

Let $\eta_0 : \pi_1(B_0, x_0) \to \pi_1(M, x_0)$ be the homomorphism induced by the inclusion orbi-map. Let $p : \widetilde{M} \to M$ be the covering associated with $\eta_0 \pi_1(B_0, x_0)$. Let $\tilde{\alpha}$ be a lifting of α with $\tilde{\alpha}(0) = \tilde{x}_0$, let $\tilde{x}_1 = \tilde{\alpha}(1)$ and let $\widetilde{B}_i$ be the component of $p^{-1}(B_i)$ with $x_i \in |\widetilde{B}_i|$.

Let $\eta_1 : \pi_1(B_1, x_1) \to \pi_1(M, x_1)$, $\tilde{\eta}_i : \pi_1(\widetilde{B}_i, \tilde{x}_i) \to \pi_1(\widetilde{M}, \tilde{x}_i)$ be the homomorphisms induced by the inclusion orbi-maps. Let $\Psi_\alpha : \pi_1(M, x_1) \to \pi_1(M, x_0)$ and $\Psi_{\tilde{\alpha}} : \pi_1(\widetilde{M}, \tilde{x}_1) \to \pi_1(\widetilde{M}, \tilde{x}_0)$ be the change of base point maps. By the same way as in the proof of proof of 13.6 of [H], we can show that $p_* \Psi_{\tilde{\alpha}} \tilde{\eta}_1 \pi_1(\widetilde{B}_1, \tilde{x}_1) = \Psi_\alpha \eta_1 \pi_1(B_1, x_1) \cap \eta_0 \pi_1(B_0, x_0)$ and $\Psi_\alpha \eta_1 \pi_1(B_1, x_1) \cap \eta_0 \pi_1(B_0, x_0)$ has finite index in each term. (See pp. 144–145 of [H].) Then, we can conclude that $\widetilde{B}_1$ is compact (note that $\widetilde{B}_0$ is already compact, since $p \mid \widetilde{B}_0$ is an isomorphism).

Case 1. There is no orbi-homotopy whose underlying map retracts (rel x_0, x_1) α into $|\partial M| - \Sigma M$ (this is certainly the case if $B_0 \neq B_1$).

With this assumption, we must have $\widetilde{B}_0 \neq \widetilde{B}_1$; otherwise, since $\tilde{\eta}_0 : \pi_1(\widetilde{B}_0) \to \pi_1(\widetilde{M})$ (the homomorphism induced by the inclusion orbi-map) is epic, there is a path $\tilde{\beta}$ in $|\widetilde{B}_0| - p^{-1}(\Sigma B_0)$ from $\tilde{x}_1$ to $\tilde{x}_0$ such that $[\tilde{\beta} \circ \tilde{\alpha}] = 1$ in $\pi_1(\widetilde{M})$. Contradiction.

In addition, we can conclude that $\widetilde{B}_0$ is incompressible in $\widetilde{M}$; if not, there is a compressing discal orbifold $D^2(n)$ with $\partial D^2(n) \subset \widetilde{B}_0$. Let M_i be a component (possibly one) of $\mathrm{cl}(\widetilde{M} - D^2(n) \times I)$. Then, by 3.9 and Van Kampen's Theorem, we have either $\pi_1(\widetilde{M}) = \pi_1(M_1) *_{\pi_1(D^2(n))} \pi_1(M_2)$ (nontrivial amalgamated free product), or $\pi_1(M_1) *_{\pi_1(D^2(n))} \mathbf{Z}$ (HNN group). Since $|\widetilde{B}_1| \cap |D^2(n)| = \varnothing$, either $\widetilde{B}_1 \subset M_1$ or $\widetilde{B}_1 \subset M_2$. Since $\pi_1(M_i) \supsetneqq \pi_1(D^2(n))$ $(i = 1, 2)$, by 4.4.1 and 4.4.2 of §4.2 of [M-K-S], $\pi_1(\widetilde{B}_1)$ has infinite index in $\pi_1(\widetilde{M})$.

On the other hand,

$$\begin{aligned}
&|\pi_1(\widetilde{M}, x_0); \psi_{\tilde{\alpha}}\tilde{\eta}_1\pi_1(\widetilde{B}_1, \tilde{x}_1)| \\
&\quad = |p_*\pi_1(\widetilde{M}, x_0); p_*\Psi_{\tilde{\alpha}}\tilde{\eta}_1\pi_1(\widetilde{B}_1, \tilde{x}_1)| \\
&\quad = |\eta_0\pi_1(B_0, x_0); \Psi_\alpha\eta_1\pi_1(B_1, x_1) \cap \eta_0\pi_1(B_0, x_0)| < \infty .
\end{aligned}$$

Contradiction.

Then B_0 is incompressible in M. Hence, $\eta_0 : \pi_1(B_0, x_0) \to \pi_1(M, x_0)$ is monic. Thus, $\tilde{\eta}_0 : \pi_1(\widetilde{B}_0, \tilde{x}_0) \to \pi_1(\widetilde{M}, \tilde{x}_0)$ is an isomorphism.

Since $\tilde{\eta}_0\pi_1(\widetilde{B}_0, \tilde{x}_0) = \pi_1(\widetilde{M}, \tilde{x}_0)$ is a fundamental group of a closed 2-orbifold, there is a torsion free, finite index, normal subgroup G of $\pi_1(\widetilde{M}, \tilde{x}_0)$.

Let $p' : (M', x'_0) \to (\widetilde{M}, \tilde{x}_0)$ be the covering associated with G. Note that $p' : M' \to \widetilde{M}$ is a finite uniformization of $\widetilde{M}$. Hence, each component of $p'^{-1}(\widetilde{B}_0)$ and $p'^{-1}(\widetilde{B}_1)$ is compact.

Let B'_0 be a component of $p'^{-1}(\widetilde{B}_0)$ and x'_0 a lift of $\tilde{x}_0$ in B'_0. Let α' be a lift of $\tilde{\alpha}$ with $\alpha'(0) = x'_0$. Let B'_1 be the component of $p'^{-1}(\widetilde{B}_1)$ with $x'_1 = \alpha'(1) \in |B'_1|$. Since $\widetilde{B}_0 \neq \widetilde{B}_1$, $B'_0 \neq B'_1$. Since $\tilde{\eta}_0$ is monic, the homomorphism induced by the inclusion orbi-map $\eta'_0 : \pi_1(B'_0, x'_0) \to \pi_1(M', x'_0)$ is monic.

Let $\tilde{p}' : \widetilde{M}' \to M'$ be the covering associated with $\eta'_0\pi_1(B'_0, x'_0)$. In the same way as in the proof of 13.6 of [H] (see pp. 144–145 of [H]), we can conclude that $\widetilde{M}'$ is compact. Hence $\widetilde{M}$ is compact. This implies that $\eta_0\pi_1(B_0)$ has finite index in $\pi_1(M)$. By 6.3, M is an I-bundle over a closed 2-orbifold.

In case $B_0 \neq B_1$, $|\pi_1(M); \eta_0\pi_1(B_0)| = 1$. In case $B_0 = B_1$, since $(f' \mid B_0) : B_0 \to C$ must not be an orbi-isomorphism from the hypothesis,

$$|\pi_1(C); (f' \mid B_0)_*\pi_1(B_0)| \geq 2 .$$

On the other hand,

$$\begin{aligned}
2 &\geq |\pi_1(M); \eta_0\pi_1(B_0)| \\
&= |\pi_1(N'); f'_*\eta_0\pi_1(B_0)| \\
&= |\pi_1(N'); i_*(f' \mid B_0)_*\pi_1(B_0)| \\
&= |\pi_1(N'); i_*\pi_1(C)| \times |\pi_1(C); (f' \mid B_0)_*\pi_1(B_0)| ,
\end{aligned}$$

where $i : C \to N'$ is the inclusion orbi-map. (Note that C is incompressible in N', since f'_* is an isomorphism, $f' \mid B_0$ is an orbi-covering, and B_0 is incompressible in M.) Hence, in any case, $|\pi_1(N'); i_*\pi_1(C)| = 1$. Thus, by 6.1, $N' = C \times [0, 1]$ $(C = C \times 0)$.

Define an orbi-map $\rho_t : C \times [0, 1] \to C \times [0, 1]$ by $\rho_t = \mathrm{id} \times c_t$, where $c_t : [0, 1] \to [0, 1]$ is the map defined by $c_t(s) = -st + s$. (Note that we regard c_t as the orbi-map both of whose underlying map and structure map are the map $s \to -st + s$.) Since $\rho_0 = \mathrm{id}$ and $\rho_1(C \times [0, 1]) \subset C \times 0$, $f_t = q \circ \rho_t \circ f'$ is the desired orbi-homotopy.

Case 2. There is an orbi-homotopy whose underlying map retracts $(\mathrm{rel}\, x_0, x_1)\alpha$ into $|\partial M| - \Sigma M$. (Automatically, $B_0 = B_1$.)

Let α_1 be a path in $|B_0| - \Sigma B_0$ such that there is an orbi-homotopy in M whose underlying map takes $(\mathrm{rel}\, x_0, x_1)$ α to α_1. Since the loop $\bar{f}' \circ \alpha_1$ lifts to a path α_1 (not a loop) under the covering $(\bar{f}' \mid B_0) : B_0 \to C$, $[\tilde{f}' \circ \hat{\alpha}_1] \neq 1$ in $\pi_1(C, y)$, where $\hat{\alpha}_1$ is a lift of α_1 in the universal covering of M. On the other hand, $[\tilde{f}' \circ \hat{\alpha}_1] = [\tilde{f}' \circ \hat{\alpha}]^{-1} \cdot [\tilde{f}' \circ \hat{\alpha}_1] = [\tilde{f}' \circ (\hat{\alpha}^{-1} \cdot \hat{\alpha}_1)] = f'_*[\hat{\alpha}^{-1} \cdot \hat{\alpha}_1] = f'_*(1) = 1$ in $\pi_1(N', y)$. Thus, C is compressible in N'.

Let $\bar{p} : \overline{M} \to M$ be a finite uniformization. Put $G = \bar{p}_*\pi_1(\overline{M})$. Let $\bar{q} : \overline{N} \to N'$ be the covering associated with $f'_*(G)$. Since G is torsion free, $f'_*(G)$ is also torsion free. Since $|\pi_1(N'); f'_*(G)| = |f'_*\pi_1(M); f'_*(G)| = |\pi_1(M); G| < \infty$, $\bar{q} : \overline{N} \to N'$ is a finite uniformization.

Let $\hat{f} : \overline{M} \to \overline{N}$ be a lift of the composition orbi-map of f' and the orbi-covering with underlying map $\bar{p}$ with respect to $\bar{q}$. It is easy to see that $\hat{f}_*$ is an isomorphism and $\hat{f}$ is a covering on each component of $\partial \overline{M}$.

If $\hat{f}$ maps two distinct components of $\partial \overline{M}$ to the same component of $\partial \overline{N}$, then there is a path β in $\overline{M} - \bar{p}^{-1}(\Sigma M)$ joining these components such that $\bar{\hat{f}} \circ \beta$ is a loop. Since $\hat{f}_*$ is epic, we may assume $[\tilde{\hat{f}} \circ \tilde{\beta}] = 1$ in $\pi_1(\overline{N})$, where $\tilde{\beta}$ is a lift of β into the underlying space of the universal covering orbifold of $\overline{M}$, $\bar{\hat{f}}$ and $\tilde{\hat{f}}$ are the underlying map and structure map of $\hat{f}$, respectively. (Note that we may assume that $\tilde{\hat{f}} = \hat{f}$.) Then $\bar{p} \circ \beta$ satisfies $(*)$ and there is no orbi-homotopy $(\mathrm{rel}(\bar{p} \circ \beta)(0), (\bar{p} \circ \beta)(1))$ whose underlying map takes $\bar{p} \circ \beta$ into $|\partial M| - \Sigma M$. Hence, we are back to the previous case. So we may suppose that $\hat{f}$ takes distinct components of $\partial \overline{M}$ to distinct components of $\partial \overline{N}$.

If $\overline{B}_0$ is a component of $\bar{p}^{-1}(B_0)$, then $\hat{f} \mid \overline{B}_0$ is not one to one. This is because a lifting $\bar{\alpha}$ of α which begins in $\overline{B}_0$ must also end in $\overline{B}_0$ whereas $\bar{f}' \circ \hat{\alpha}$ is a loop (hence, so is $\hat{f} \circ \bar{\alpha}$ in $\overline{N}$) in the underlying space of the universal covering orbifold of N'.

By the same way as in the proof of 13.6 of [H] (see pp. 146–147 of [H]), we can conclude $\chi(\hat{f}(\overline{B}_0)) = 0$. Note that $\bar{q}$ can be restricted to a covering from $\hat{f}(\overline{B}_0)$ to C. Let k be the number of sheets of $(\bar{q} \mid \hat{f}(\overline{B}_0)) : \hat{f}(\overline{B}_0) \to C$. Since $\chi(C) = (1/k)\chi(\hat{f}(\overline{B}_0)) = 0$, C is either a torus or $S^2(2, 2, 2, 2)$.

Since $\chi(q(C)) = 0$, $q(C)$ is also either a torus or $S^2(2, 2, 2, 2)$. $q(C)$ is compressible in N; otherwise, by 3.8, C is incompressible in N'. Hence N is either (a) or (b) in (3).

Since $f: M \to N$ can be restricted to an orbi-covering on each component of ∂M, B_0 is either a torus or $S^2(2, 2, 2, 2)$. Since $(f \mid B_0): B_0 \to q(C)$ is an orbi-covering, by 3.4, B_0 is compressible in M.

Hence M is either (a) or (b) in (3). Thus, we obtain conclusion (3).

In case $f' \mid \partial M$ *is an orbi-embedding* (*automatic if* $|\partial M| = \varnothing$). Let $N' = N'_0 \supset N'_1 \supset \cdots \supset N'_m$ be a hierarchy for N'. We prove the following Proposition (n).

Proposition (n). *Suppose* $f' : (M, \partial M) \to (N', \partial N')$ *is an orbi-map such that* f'_* *is monic and* $f' \mid \partial M$ *is an orbi-embedding. If* N' *possesses a hierarchy of length* n, *then there exists an orbi-map* $g' : M \to N'$ *such that*

(i) $g'_* = f'_* : \pi_1(M) \to \pi_1(N')$.
(ii) $g' : M \to N'$ *is an orbi-covering, and*
(iii) $g' \mid \partial M = f' \mid \partial M$.

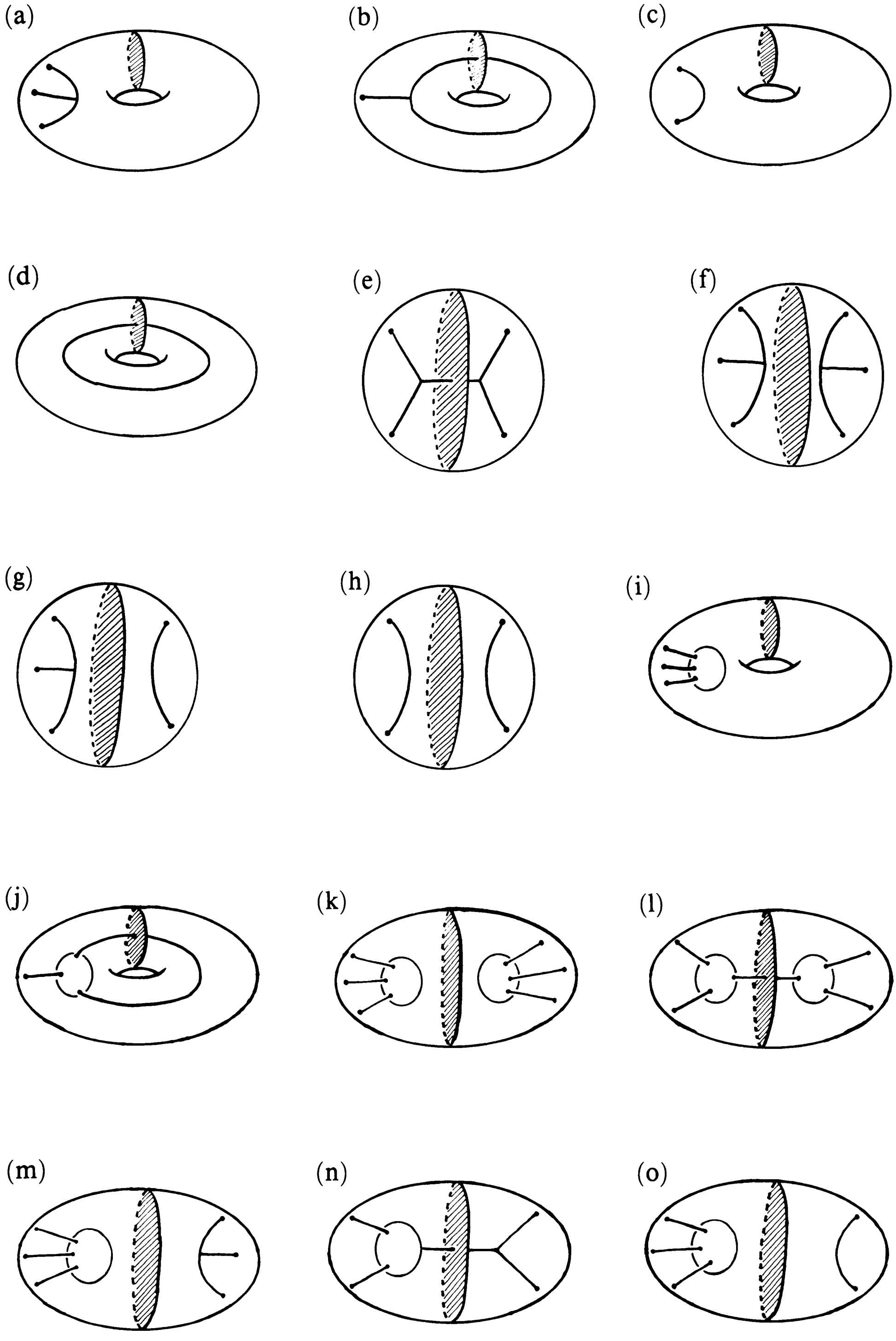
Let G be the incompressible 2-suborbifold of N'. By 5.5, we may change f' by a C-equivalent modification so that each component of $f'^{-1}(G)$ is a 2-sided and incompressible 2-suborbifold of M. Note that the C-equivalent modification except removing compressible discal orbifold does not change $f' \mid \partial M$. In addition $f'^{-1}(G)$ must not be a compressible discal orbifold; if not, G is also a compressible discal orbifold. Thus this modification does not change $f' \mid \partial M$.

Let F be a component of $f'^{-1}(G)$. We have $\overline{f'}(|F|) \cap (|G| - \Sigma G) \neq \varnothing$. (In case $|\partial G| \neq \varnothing$, it is clear since $f' \mid \partial M$ is an orbi-embedding. In case $|\partial G| = \varnothing$, it is shown by using the fact that F is not spherical, incompressible in M, and f'_* is monic.) So we can define the restriction orbi-map $f' \mid F$. Since F is incompressible and f'_* is monic, $(f' \mid F)_* : \pi_1(F) \to \pi_1(G)$ is monic.

If $n = 1$, then N' is either a solid torus or one of (a)–(h) in Figure 7.3. Let $D^2(r)$ be the incompressible discal orbifold indicated in Figure 7.3. Since M is irreducible and sufficiently large, we may assume that $f'^{-1}(D^2(r))$ consists of at least one incompressible discal orbifold properly embedded in M. Namely, $\partial M \neq \varnothing$. In any case of (a)–(h), since the number of the components of $\partial N'$ is one and $f \mid \partial M$ is orbi-embedding, ∂M is orbi-isomorphic to $\partial N'$. Let S be the boundary of $D^2(r)$. Since f'_* is monic, by 3.4, $f'^{-1}(S)$ also bounds the discal orbifold $D^2(r)$. By the irreducibility of M, we can easily extend the orbi-isomorphism $(f' \mid \partial M) : \partial M \to \partial N'$ to an orbi-isomorphism $g' : M \to N'$. Since $g' \mid \partial M = f' \mid \partial M$, $g'_* = f'_*$.

Suppose that $n > 1$. Let G be the incompressible 2-suborbifold of N' along which we cut to obtain N'_1.

Suppose $|\partial G| \neq \varnothing$. Since the fundamental group of a closed nonspherical 2-orbifold must not be a subgroup of the fundamental group of a 2-orbifold



Shadowed discs indicate incompressible discal orbifolds

FIGURE 7.3

with boundaries, each component of $f'^{-1}(G)$ has boundaries. Note that

$$(f' \mid f'^{-1}(G)) : f'^{-1}(G) \to G$$

is boundary preserving and this is an orbi-embedding on $\partial f'^{-1}(G)$. If G is $D^2(r)$, then, since the underlying map of $f' \mid (\partial f'^{-1}(G))$ is a homeomorphism, $f'^{-1}(G)$ is also $D^2(r)$. So, by 7.1, we can modify $f' \mid f'^{-1}(G)$ to an orbi-embedding under an orbi-homotopy without changing $f' \mid \partial(f'^{-1}(G))$. In any other case 7.2 applies; so we may modify $f' \mid f'^{-1}(G)$ to an orbi-covering under an orbi-homotopy without changing $f' \mid \partial(f'^{-1}(G))$. (Furthermore, since $f' \mid \partial f'^{-1}(G)$ is an orbi-embedding, the orbi-covering is an orbi-embedding (1-sheeted orbi-covering).) The (2) of 7.2 does not occur since $f' \mid \partial(f'^{-1}(G))$ is an orbi-embedding. For the same reason, we must have (the modified) $f' \mid f'^{-1}(G)$ an orbi-covering if $|\partial G| = \emptyset$.

Let Q be a component of N_1'. By the above, we may assume that if P is a component of $f'^{-1}(Q)$, then $f' : P \to Q$ is boundary preserving and f' is an orbi-covering on each component of ∂P. By incompressibility, $(f' \mid P)_* : \pi_1(P) \to \pi_1(Q)$ is monic. Note that Q possesses a hierarchy of length not more than $n-1$.

If $f' \mid \partial P$ is an orbi-embedding (which must occur if $|\partial G| \neq \emptyset$), then we apply induction hypothesis to get an orbi-map satisfying (i)–(iii).

If $f' \mid \partial P$ is not an orbi-embedding (so $|\partial G| = \emptyset$), then our initial considerations apply to show that P is an I-bundle over a closed 2-orbifold and that $f' \mid P$ orbi-homotopes into G without changing $f' \mid \partial P$. Note that conclusion (3) does not occur here, since P contains a closed incompressible 2-suborbifold (parallel to a component of $f'^{-1}(G)$) in its boundary. On the other hand, any-one of (3) does not contain such a 2-suborbifold. Since $P \subset \mathrm{Int}(M)$, we can eliminate (one or two) components of $f'^{-1}(G)$ by modifying f' under an orbi-homotopy without changing $f' \mid \partial M$. By continuing in this manner, we must arrive at a point where f' is an orbi-embedding on the boundary of each component of $f'^{-1}(N_1')$. We cannot completely eliminate $f'^{-1}(G)$; otherwise, by induction hypothesis, we can change $f' : M \to Q$ into an orbi-covering h' satisfying $h'_* = f'_*$ without changing $f' \mid \partial M$. Since $G \subset \partial Q$ and h' is an orbi-covering, $|h'(\partial M)| \cap |G| \neq \emptyset$. Since $(f' \mid \partial M) = (h' \mid \partial M)$ and $G \subset \mathrm{Int}(N')$, $|f'(\partial M)| \cap |\mathrm{Int}(N')| \neq \emptyset$. This contradicts the fact that $f'(\partial M) \subset \partial N'$.

Applying induction and piecing together the results, we get an orbi-map g' which satisfies (i)–(iii). Hence $g = q \circ g'$ is the desired orbi-map. Q.E.D.

Let M and N be 3-orbifolds. Let $\Psi : \pi_1(M, x) \to \pi_1(N, y)$ be a homomorphism. We say that Ψ *respects the peripheral structure*, if the following holds. For each boundary component F of M, there exists a boundary component G of N, such that $\Psi(i_*(\pi_1(F, x'))) \subset A$, and A is conjugate in $\pi_1(N, y)$ to $j_*(\pi_1(G, y'))$, where i and j are inclusions.

7.5. **Lemma.** *Let M and N be compact and orientable 3-orbifolds, such that* *each component of ∂N is incompressible and that the underlying space of the universal covering orbifold of* $\operatorname{Int}(N)$ *is homeomorphic to* $\mathbf{R}^3$. *Let* $\Psi : \pi_1(M, x) \to \pi_1(N, y)$ *be a homomorphism which respects the peripheral structure. Then, there exists an orbi-map* $f : (M, \partial M) \to (N, \partial N)$ *which induces* Ψ *and for each component* B *of* ∂M, $f \mid B$ *can be defined.*

Proof. By 4.2, there is an orbi-map $f' : M \to N$ which induces Ψ. To prove the lemma, it will suffice to show that if F is a component of ∂M, then there exists an orbi-homotopy $H : F \times [0, 1] \to N$ such that $H \mid (F \times 0) = f' \mid F$ and $H \mid (F \times 1)$ is an orbi-map into ∂N. We construct this orbi-homotopy piecewise. Define $H \mid (F \times 0) = f' \mid F$. Take a triangulation K_F of $|F|$ so that for each 2-simplex $e \in K_F$, $\partial e \cap \Sigma F = \varnothing$ and $(\operatorname{Int}(e)) \cap \Sigma F =$ (at most one point). Let F_1 be the subspace of F whose underlying space is $|K_F^{(1)}| \times |[0, 1]|$. From the hypothesis that f' respects the peripheral structure, we can extend $H \mid (F \times 0)$ to F_1 with $\overline{H}(|K_F^{(1)}| \times 1) \subset |\partial N| - \Sigma N$. Let G be the component of ∂N such that $\overline{H}(|K_F^{(1)}| \times 1) \subset |G|$. Since $\operatorname{Ker}(i_* : \pi_1(G) \to \pi_1(N)) = 1$, we can extend $H \mid \{(F \times 0) \cup F_1\}$ to $F \times 1$. By 4.1, we can extend $H \mid \{(F \times 0) \cup (F \times 1) \cup F_1\}$ to the rest. Q.E.D.

Note that, in 7.4, the condition that $(f \mid B)_* : \pi_1(B) \to \pi_1(C)$ is monic is automatically satisfied if all the components of ∂M are incompressible in M (or if $|\partial M| = \varnothing$).

7.6. **Theorem.** *Let* $M, N \in \mathscr{W}$. *Suppose all the components of* ∂M *and* ∂N *are incompressible in* M. *Let* $\Psi : \pi_1(M) \to \pi_1(N)$ *be an isomorphism which respects the peripheral structure. Then either*

(1) *there exists an orbi-isomorphism* $f : M \to N$ *which induces* Ψ *or*

(2) M *is a twisted* I*-bundle over a closed nonorientable* 2*-orbifold* F *and* N *is an* I*-bundle over a* 2*-orbifold* G *such that* $\pi_1(F) \cong \pi_1(G)$.

Proof. By 7.5, we can construct an orbi-map $f' : (M, \partial M) \to (N, \partial N)$ such that $f'_* = \Psi$ and for each component B of ∂M, $f' \mid B$ can be defined. If M is not (2) of 7.4, we apply 7.4 to obtain a 1-sheeted orbi-covering (i.e. an orbi-isomorphism). Note that (3) of 7.4 must not occur since each component of ∂M is incompressible in M.

Suppose M is an I-bundle over a closed 2-orbifold F. By 6.3, N is an I-bundle over a closed 2-orbifold G such that $\pi_1(F) \cong \pi_1(G)$. If F is orientable, then $M = F \times I$. In addition, $N = G \times I$. Otherwise, $f'_* \pi_1(M)$ is contained in $\pi_1(\partial N)$ which has index two in $\pi_1(N)$. Let F' be a component of ∂M and G' a component of ∂N such that $f'(F') \subset G'$. Since $(f' \mid F')_* : \pi_1(F') \to \pi_1(G')$ is an isomorphism, by 7.2, we can modify $f' \mid F'$, under an orbi-homotopy, to an orbi-isomorphism $g : F' \to G'$ such that $g_* = (f' \mid F')_*$. Define an orbi-isomorphism $f : F \times I \to G \times I$ by $f = g \times \mathrm{id}$. It is clear that $f_* = \Psi$. Q.E.D.

7.7. **Corollary.** *Let* $M, N \in \mathscr{W}$. *Suppose* M *and* N *are closed and* $\pi_1(M) \cong \pi_1(N)$. *Then* M *and* N *are orbi-isomorphic.*

8. Applications to links and tangles

Let X be an orbifold such that $\Sigma X = K_1 \cup \cdots \cup K_r$ and the order of the local group of any point in K_i is n, $i = 1, 2, \ldots, r$, where K_i's are disjoint smooth simple closed curves. Let $p : M \to S^3$ be the n-fold cyclic branched covering branched over L. Clearly $p : M \to S^3$ is a finite uniformization of X and X is compact, connected, orientable, and contains no turnovers. Recall that a link L of disjoint simple closed curves in S^3 is *prime* if there is no S^2 in S^3 that separates the component of L, and any S^2 that meets L in two points, transversely, bounds in S^3 one and only one ball intersecting L in a single unknotted spanning arc. Hence, if $K_1, \ldots, K_r$ is a prime link in S^3, X is irreducible. We call the link $\{K_1, \ldots, K_r\}$ in S^3 *sufficiently large* if X is sufficiently large. This definition depends only on the link type of $\{K_1, \ldots, K_r\}$ not on the natural numbers $n \geq 2$.

Let (S^3, L) be a link and X be the orbifold which satisfies that $\Sigma X = L$ and the orders of every local group of ΣX are $n \in \mathbf{Z}$, $n \geq 2$. We call such an orbifold X *the associated orbifold with weight* n *of* (S^3, L), denoted by $O(L, n)$. We define the *n-weighted orbi-invariant* of (S^3, L), denoted by $\mathrm{Orb}_n(L)$, by the fundamental group of the associated orbifold with weight n of (S^3, L). Clearly, if two links (S^3, L) and (S^3, L') are the same link type, then $\mathrm{Orb}_n(L) \cong \mathrm{Orb}_n(L')$ for any $n \in \mathbf{N}$. If (S^3, L) and (S^3, L') are prime and sufficiently large, then the converse holds.

8.1. **Theorem.** *Suppose (S^3, L) and (S^3, L') are prime and sufficiently large links. (S^3, L) and (S^3, L') are the same link type, if and only if* $\mathrm{Orb}_n(L) \cong \mathrm{Orb}_n(L')$ *for some $n \in \mathbf{Z}$, $n \geq 2$.*

Proof. One direction is obvious. We come to the other. Let $O_{(L,n)}$ and $O_{(L',n)}$ be the associated orbifolds with weight n of (S^3, L) and (S^3, L'), respectively. Since $\pi_1(O_{(L,n)}) \cong \pi_1(O_{L',n)})$, by 7.7, $O_{(L,n)}$ and $O_{(L',n)}$ are orbi-isomorphic. The orbi-isomorphism is a homeomorphism from S^3 to S^3 and carries L to L'. Q.E.D.

We give a sufficient condition so that a link (S^3, L) is sufficiently large.

A k-strings *tangle* is a pair (B, t) where B is a 3-ball and t is k disjoint arcs in B with $t \cap \partial B = \partial t$. Two tangles (B_1, t_1) and (B_2, t_2) are *equivalent* if there is a homeomorphism of pairs from (B_1, t_1) to (B_2, t_2). A tangle (B, t) is *trivial* if it is equivalent to $(D^2 \times I, \{x_i\} \times I)$, where x_i's are disjoint points in $\mathrm{Int}(D^2)$. We define *the associated orbifold with weight* n, denoted by $O_{(t,n)}$, in a manner similar to the case of links.

8.2. **Proposition.** *Let (S^3, L) be a prime link. If there is a 2-strings prime tangle (B, t) such that $O_{(L,n)} - \mathrm{Int}(O_{(t,n)})$ does not contain any properly embedded separating disc, then (S^3, L) is sufficiently large.*

Proof. It is clear that $\partial O_{(t,n)}$ is an incompressible 2-suborbifold in $O_{(L,n)}$. Q.E.D.

We define the n-weighted orbi-invariant of a tangle (B, t), denoted by $\mathrm{Orb}_n(t)$, by the fundamental group of the associated orbifold with weight n of (B, t). As an application of 4.2 and 5.5, we can get the following "Untangling theorem."

8.3. **Theorem.** *Let (B, t) be a tangle. (B, t) is the k-strings trivial tangle if and only if* $\mathrm{Orb}_2(t) \cong A_1 * \cdots * A_k$, *where* $A_i \cong \mathbf{Z}_2$ *for each* i.

Proof. Let (B, t') be the k-strings trivial tangle. Note that $\pi_1(O(t', 2)) \cong A_1 * \cdots * A_k$, where $A_i \cong \mathbf{Z}_2$ for each i. By the hypothesis and 4.2, we can construct an orbi-map $f : O(t, 2) \to O(t', 2)$ such that $f_* : \pi_1(O(t, 2)) \to \pi_1(O(t', 2))$ is an isomorphism. Let D_i, $i = 1, 2, \ldots, k-1$, be a mutually disjoint, properly embedded, incompressible discs in $O(t', 2)$ such that each component of $\mathrm{cl}(O(t', 2) - \bigcup(D_i \times I))$ contains one and only one singular locus. By 5.5, we have a sequence $O(t, 2) = M_1 \supset M_2 \supset \cdots \supset M_n$ such that M_{i+1} is obtained from M_i by cutting open along a properly embedded disc and there is an orbi-map from each component of M_n into a component of $\mathrm{cl}(O(t', 2) - \bigcup(D_i \times I))$ which induces a monomorphism between the fundamental groups. Since the fundamental group of each component of $\mathrm{cl}(O(t', 2) - \bigcup(D_i \times I))$ is $\mathbf{Z}_2$, each component of M_n is a ballic orbifold of cyclical type. Then the conclusion holds. Q.E.D.

By Theorem 11 of [D], the orbifold which belongs to $\mathscr{W}$ is sufficiently large if it has a boundary component which is not a turnover. So we can get the following.

8.4. **Proposition.** *Suppose (B_1, t_1) and (B_2, t_2) are tangles which contain no local knots. (B_1, t_1) and (B_2, t_2) are equivalent, if and only if, for some $n \in \mathbf{Z}$, $n \geq 2$, there is an isomorphism* $\mathrm{Orb}_n(t_1) \cong \mathrm{Orb}_n(t_2)$ *which respects the peripheral structure.*

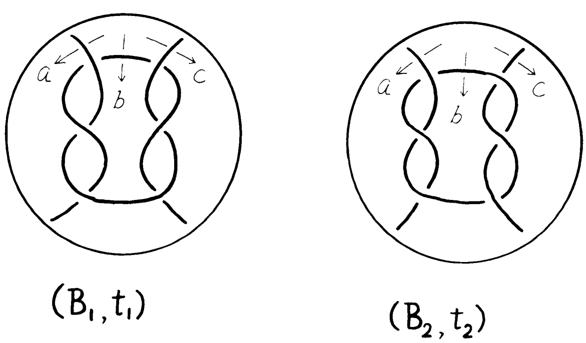


FIGURE 8.1

8.5. *Remark.* The condition "respects the peripheral structure" is necessary. Let (B_1, t_1) and (B_2, t_2) be tangles in Figure 8.1.

$$\mathrm{Orb}_2(t_1) \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, (ab)^2 a = (cb)^2 c \rangle$$

and

$$\mathrm{Orb}_2(t_2) \cong \langle a, b, c \mid a^2 = b^2 = c^2 = 1, (ab)^2 a = (bc)^3 b \rangle.$$

There is an isomorphism defined by $a \to a$, $b \to b$, $c \to bcb$. But, clearly (B_1, t_1) and (B_2, t_2) are not equivalent. Note that the isomorphism does not respects the peripheral structure.

References

[B-K-S] N. P. Buchdahl, S. Kwasik, and R. Schultz, *One fixed point actions on low-dimensional spheres*, Invent. Math. **102** (1990), 633–662.

[B-N] S. Bundgaard and J. Nielsen, *On normal subgroups with finite index in F-groups*, Math. Tidsskr. B **1951**, 56–58.

[B-S 1] F. Bonahon and L. Siebenmann, *The classification of Seifert fibered 3-orbifolds*, Lecture Notes Ser., no. 95, London Math. Soc., 1985, pp. 19–85.

[B-S 2] ——, *The characteristic toric splitting of irreducible compact 3-orbifolds*, Math. Ann. **278** (1987), 441–479.

[B-Z] M. Boileau and B. Zimmermann, *The π-orbifold group of a link*, Math. Z. **200** (1989), 187–208.

[D] W. D. Dunbar, *Hierarchies for 3-orbifolds*, Topology Appl. **29** (1988), 267–283.

[F] R. H. Fox, *Fenchel's conjecture about F-groups*, Math. Tidsskr. B **1952**, 61–65.

[H] J. Hempel, *3-manifolds*, Ann. of Math. Studies, no. 86, Princeton Univ. Press, 1976.

[K] M. Kato, *On uniformization of orbifolds*, Adv. Studies in Pure Math. **9** (1986), 149–172.

[M-K-S] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*, Wiley, New York, 1966.

[M-M] D. McCullough and A. Miller, *Manifold covers of 3-orbifolds with geometric pieces*, Topology Appl. **31** (1989), 169–185.

[M-S] W. H. Meeks and P. Scott, *Finite group actions on 3-manifolds*, Invent. Math. **86** (1986), 287–346.

[M-Y 1] W. H. Meeks and S. T. Yau, *Topology of three dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2) **112** (1980), 441–484.

[M-Y 2] ——, *The equivariant Dehn's lemma and loop theorem*, Comment. Math. Helv. **56** (1981), 225–239.

[M-Y 3] ——, *Embedded minimal surfaces, exotic spheres, and manifolds with positive Ricci curvature*, Ann. of Math. (2) **116** (1982), 621–659.

[M-Y 4] ——, *Group actions on* $\mathbf{R}^3$, The Smith Conjecture, Academic Press, New York, 1984, pp. 169–179.

[S] I. Satake, *On a generalization of the notion of manifold*, Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 359–363.

[Ta 1] Y. Takeuchi, *A classification of a class of 3-branchfolds*, Trans. Amer. Math. Soc. **307** (1988), 481–502.

[Ta 2] ——, *The untangling theorem*, Topology Appl. **34** (1990), 129–137.

[Th] W. P. Thurston, *The geometry and topology of three-manifolds*, mimeographed notes, Princeton Univ., 1978.

[W] F. Waldhausen, *On irreducible 3-manifolds which are sufficiently large*, Ann. of Math. (2) **87** (1968), 56–88.

[Y] M. Yamasaki, *Maps between orbifolds*, Proc. Amer. Math. Soc. **109** (1990), 223–232.

[Z] B. Zimmermann, *Some groups which classify knots*, Math. Proc. Cambridge Philos. Soc. **104** (1988), 417–418.

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